GEOMETRY FOR SCHOOLS

THEORETICAL

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GEOMETRY FOR SCHOOLS

THEORETICAL

A SEQUEL TO—
ELEMENTARY PLANE GEOMETRY
INDUCTIVE AND DEDUCTIVE

BY

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PREFACE.

THE question of improvement in the teaching of elementary Geometry has been so long under consideration and discussion, that agreement in the main has been reached as to the lines along which reform should proceed. Thus the Report of the Committee of the British Association, the Report of the Committee of the Mathematical Association (formerly the Association for the Improvement of Geometrical Teaching), the Report of the Syndicate of the Senate of the University of Cambridge, as well as the various publications on Elementary Geometry that have recently appeared, differ in details rather than in principles. There is a general consensus of opinion on the following points:

(1). The study of formal demonstrative Geometry should be preceded by a practical course in inductive or experimental Geometry. There is general accord that this practical course should be continued during the period of elementary geometric studies. Thus the Senate of the University of Cambridge has decided that, "The paper in Geometry shall contain questions on Practical and on Theoretical Geometry. Every candidate shall be expected to answer questions in both branches of the subject."

(2). The subject should not be dissociated from other branches of mathematics—Arithmetic and Algebra—studied at the same time. This introduction of the idea of number makes possible an important change in the theory of Ratio and Proportion, and in other respects modernizes the subject. It marks of course a complete break with the spirit of Euclid.

(3). A considerable number of the propositions in Euclid are of little or no importance in the scientific development of the subject, and should be discarded. A large part of Euclid, Book II., loses significance by reason of the permissive use of algebraic forms. Much of Euclid, Book IV., is best dealt with as exercises in practical constructive Geometry. The change in the theory of Proportion makes certain propositions in Euclid, Book VI., very simple, or even unnecessary. The time thus set free may well be devoted to some more advanced branch of Geometry, *e.g.*, the elements of Analytical Geometry.

In my "Elementary Plane Geometry—Inductive and Deductive" I have sought to show how a practical course in measurement, use of simple instruments, and accurate construction may afford a training of value in itself, and also of service in anticipating the truths afterwards reached by deductive Geometry. In the present book I trust the second and third of the reforms, which teachers of mathematics have had in mind for years, and which are recommended in the Reports above referred to, may be found to have been judiciously carried out.

I have not thought it wise to separate the Problems from the Theorems. In recently published text-books on Geometry much stress is naturally laid on accuracy of construction, and it seems only fair to show how a construction may be made before directing its employment in a Theorem. In a strictly logical system of Geometry there is something to be said in favor of the Euclidean practice of avoiding as much as possible the hypothetical construction. To relegate Problems to a subordinate place is to deprive them of their due importance; and to classify Theorems and Problems separately, and then direct that they be taken up as parallel courses, is not to classify at all, but rather to create inconvenience and confusion. I have helped the learner to distinguish between Theorems and Problems by enunciating the former in black-face type and the latter in italics.

Ratio and Proportion, with Similar Triangles, have been taken up at an early period in the course. This arrangement seems fully justified: much additional power is thereby acquired, many demonstrations are simplified, and the theory of Ratio and Proportion presents no difficulty when it is not sought to include incommensurable quantities. Indeed, one of the most urgent reasons for modifying the course in Geometry is the fact that, in the past, very many pupils have left school without any knowledge of that most important problem in science,—the theory of similar triangles. Placed early in the course, it is likely to come under the notice of all.

In the Introduction certain fundamental theorems are reached which seem to flow immediately from the conception of a straight line, and from the definitions of the right angle and of parallel lines. These theorems correspond to Propositions 13, 14, 15, 16, 17 and 32 of Euclid, Book I. It would have been very easy to throw the demonstrations into the rigid form adopted in the Propositions. The student, however, encounters, in the Propositions, quite enough of the syllogistic form. Indeed, many leave school with the notion that nothing has been proved unless the proof has been arranged in the manner of Euclid's Propositions. The theorems in question are fundamental, and it was thought that their fundamental character would best be appreciated by leaving them associated with the geometrical elements and definitions from which they immediately spring.

Similarly, in developing the symmetry of the circle in the Introduction to Book IV., certain theorems have been considered which in other books rank as separate propositions. Their truth is appreciated so immediately on realizing the symmetry of the circle that it seemed well to place them in a discussion which dealt with symmetry.

In the theory of parallel lines, what may be called the directional conception of parallelism has been adopted. This seems to be more in accordance with our common notions than the negativism of Euclid's definition of parallels. It is a further advantage that a beginning be made with an exact use of the idea of direction, an idea so fruitful in various departments of mathematics. Props. 1–3 of Additional Propositions, however, offer an alternative treatment of the subject.

It is hoped that the part which the centre of similitude is made to play in dealing with the theory of similar polygons, will be felt to be a natural way of reaching results, and will be generally approved of.

The Exercises have been attached to the Propositions to which they seemed to belong, and of which they furnish applications.

Very many recommend that the definitions be given as needed. This arrangement has the effect of scattering them through the text. I have placed them in the Introduction, that they may readily be referred to when wanted. The teacher is advised to take them up as they are required.

A. B.

Toronto, September, 1904.

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GEOMETRY FOR SCHOOLS.

INTRODUCTION.

Volumes, Surfaces and Lines.

1. The **volume** of a body is the amount of space it occupies.

A volume has **length, breadth** and **thickness,** and therefore is said to be of **three dimensions.**

2. The **surface** of a body is the boundary which separates it from that part of space not occupied by the body.

A surface has **length** and **breadth,** and therefore is said to be of **two dimensions.**

The term surface is also applied to the boundary, without thickness, but with length and breadth, which separates one portion of space from another.

3. The intersections of surfaces are **lines.**

A line has length only, and therefore is said to be of **one dimension.**

The term line is also applied to the boundary, without breadth, but with length, which separates one portion of a surface from another portion of the same surface.

4. The intersections of lines are **points.**

A point has neither length, breadth nor thickness, and therefore is said to have **no dimensions.**

It has **position.**

The term point is also applied to the division, without length, which separates one portion of a line from another portion of the same line.

The ends of a line are points.

5. We may *represent* a line by a pencil mark, and a point by a dot with a pencil. We must, however, remember that these are only representations, for a dot made with a pencil has length, breadth and even thickness, inappreciable though they may be.

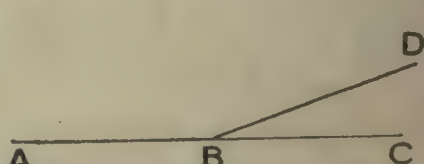
The Straight Line.

6. A **straight line** is a line which lies evenly between its extreme points.

In interpreting the words "lies evenly," we see that **a straight line is the shortest distance between its extreme points.**

7. A straight line throughout its entire length has the same **direction.** Indeed, in the ultimate analysis of our conception of direction, we reach a straight line.

Hence, **if two straight lines coincide in part, they coincide as far as the one or the other is continued.** Otherwise we should have such a result as the annexed figure presents, where ABC and ABD are both supposed to be straight lines. The parts BC and BD have different directions, and therefore cannot both have the same direction as the common part AB. Hence, each



of the lines **ABC**, **ABD** cannot have the same direction throughout its entire length, *i.e.*, they cannot both be straight lines.

8. If two straight lines intersect they have different directions, *i.e.*, they deviate from one another, and it is considered self-evident that, beyond the point of intersection, one of the lines cannot turn towards the other and again intersect it. This is expressed by saying that,—

Two straight lines can intersect in only one point.

Or,

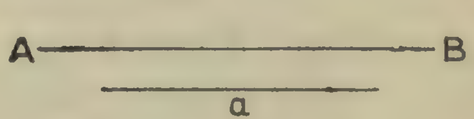
Two straight lines cannot enclose a space.

Or,

If two straight lines have two common points, then they coincide as far as the one or the other is continued.

We see also that **any straight line may be made to coincide with any other straight line as far as one or the other is continued.** For, let the lines be placed so as to intersect in **A**, and let **B** be a point in one of the lines. Let **AB** be turned about **A** until **B** falls on the other line. Since now the lines have two points common, they must coincide.

9. A line is denoted by the letters at its extremities, as the line **AB**; or by a single letter, as the line *a*.

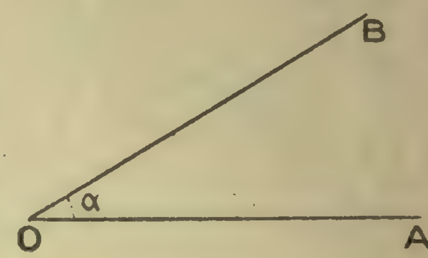


10. A plane surface, or a **plane,** is that in which any two points being taken, the straight line joining them lies wholly in that surface.

A system of geometry in which all the points and lines are supposed to lie in a plane, is called **Plane Geometry.** The present work deals with Plane Geometry only.

Angles.

11. Let a straight line rotate in a plane, say the plane of the paper, about the point **O**; and suppose that, starting from the position **OA**, it rotates into the position **OB**. The amount of turning which the line has done in rotating about **O** from the position **OA** to the position **OB**, is called an **angle.**

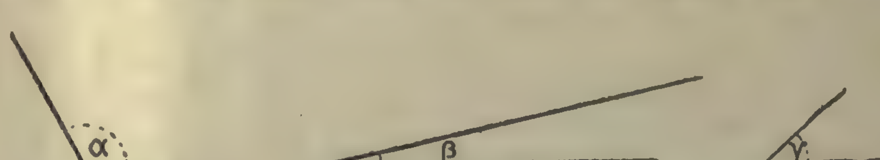


The point **O** is called the **vertex** of the angle.

The lines **OA, OB** are called the **arms** of the angle.

The angle is denoted by the three letters **AOB**, the letter at the vertex being the middle one of the three. If not more than one angle be at the point **O**, it may be denoted by the single letter **O**. The angle may also be spoken of as the angle α, the letter α being written within the angle.

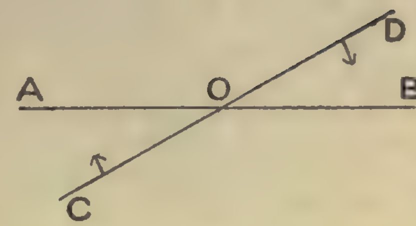
12. It is to be carefully noted that the size of the angle in no way depends on the lengths of the arms



OA, OB. Thus, of these three angles, α is the greatest, β is the least, and γ is intermediate in size.

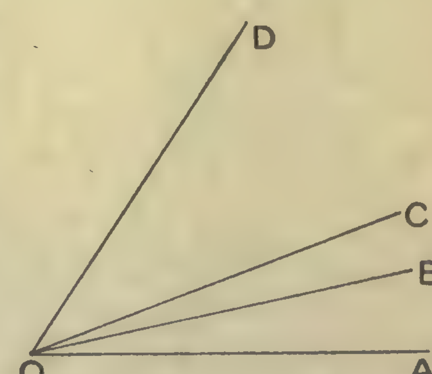
Any two lines which have different directions form an angle (or angles) at their point of intersection, and this angle measures the deviation in their directions.

13. If the lines **AOB, COD** intersect at **O**, the angles **AOC, BOD** are said to be **vertically opposite** angles. The angles **AOD, BOC** are also said to be vertically opposite angles.



Let the line **COD** rotate about **O** in the direction indicated by the arrow-heads. When the part **OC** coincides with **OA**, the part **OD** must coincide with **OB** (§ 7). Hence, the two angles **AOC, BOD**, being generated by the same amount of rotation, maintained in the same direction, must be equal. Similarly, the angles **AOD, BOC** are equal. That is,—**If two straight lines cut one another, the vertically opposite angles are equal.**

14. In the adjacent figure the angle **AOD** is equal to the sum of the angles **AOB, BOC, COD.**



The angle **COD** is equal to the difference of the angles **AOD, AOC.**

The sum of the two angles **AOC, COD** is equal to the sum of the three angles **AOB, BOC, COD.**

The angle **BOC** is common to the angles **AOC, BOD.**

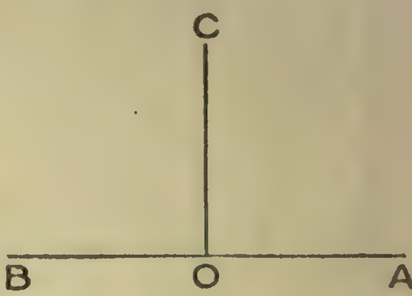
The angles **AOB, BOC** are said to be **adjacent** angles; as are also the angles **BOC, COD.**

The Right Angle.

15. When one straight line standing on another straight line makes the adjacent angles equal, each of these angles is called a **right angle,** and the straight

line which stands on the other is called a **perpendicular** to it.

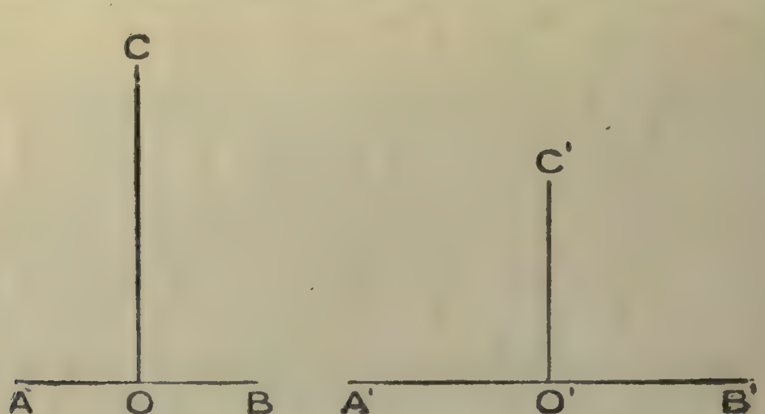
Thus, if **BOA** is a straight line, and the angles **AOC, BOC** are equal, each is a right angle.



16. If the line **OC** be supposed to start from the position **OA**, and rotate about **O** to the position **OB**, the angle **AOC** continually increases, and the angle **BOC** continually decreases. So that if these angles be once equal, they cannot again be equal, nor could they previously have been equal. Thus **only one line can be drawn from O perpendicular to** (and on the same side of) **OB**.

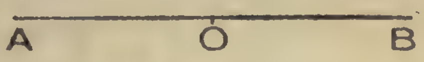
The above consideration shows that *any* angle can have only one line bisecting it. In a similar way we may show that a given line can have only one point of bisection.

17. Let **OC** be perpendicular to **AB**, and **O′C′** perpendicular to **A′B′**. Suppose the straight line **AOB** be made to coincide with **A′O′B′**, so that **O** may coincide with **O′**.

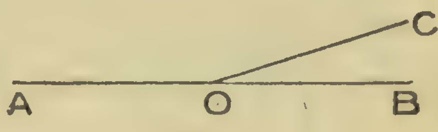


Then **OC** will coincide with **O′C′**, since, as we have just seen (§16), there can be only one line at **O′** perpendicular to A′O′B′. It follows that **all right angles are equal to one another.**

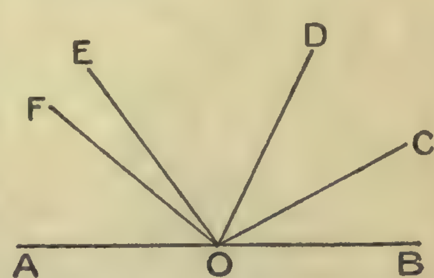
18. **The angle which one part of a straight line makes with the adjacent part,** *e.g.*, **OA** with **OB, is evidently equal to two right angles.** Such an angle is sometimes called a *straight* angle.



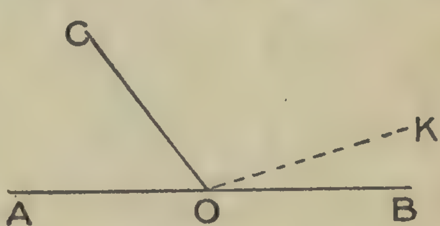
Conversely, **if OA makes with OB an angle equal to two right angles, then OA and OB are in one and the same straight line.** For, if not, let **OC** be in the same straight line with **OA**. Then the angle **AOC** is equal to two right angles, and therefore (§ 17) is equal to the angle **AOB**, which plainly is not possible.



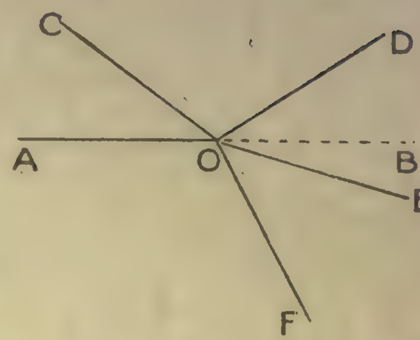
If any number of lines, **OC, OD, OE, OF,** be drawn from a point **O** in the straight line **AB**, such lines being on the same side of **AB, the sum of the angles BOC, COD, DOE, EOF, FOA is evidently equal to two right angles.**



Conversely, **if the sum of any number of such adjacent angles be two right angles, the first and final bounding lines are in one and the same straight line.** For if the angles **BOC, COA** be together equal to two right angles, and **OB** be not in the same straight line with **OA,** let **OK** be in the same straight line with **OA**. Then the angles **AOB, AOK,** being both equal to two right angles, are equal to one another, which is evidently impossible.



Since, if **AOB** be a straight line, **OA** makes with **OB** an angle equal to two right angles, on whichever side of **AOB** we regard the angle as formed, it follows that **all the angles at a point,** as **AOC, COD, DOE, EOF, FOA, are together equal to four right angles.**



19. That we may measure angles and express their magnitudes numerically, the right angle is divided into 90 equal angles called degrees; each degree is divided into 60 equal angles called minutes, and each minute is divided into 60 equal angles called seconds.

An angle of 35 degrees, 47 minutes, 23 seconds, is written 35° 47′ 23″.

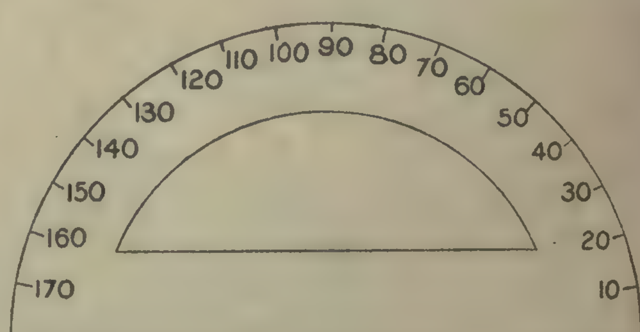
Thus the right angle is expressed by 90°; the *straight* angle by 180°; and the entire angular interval at a point by 360°.

If a line make two complete revolutions about its end (or about any point in the line), it may be said to have generated an angle of 720°. There is thus no limit to the magnitude an angle may have.

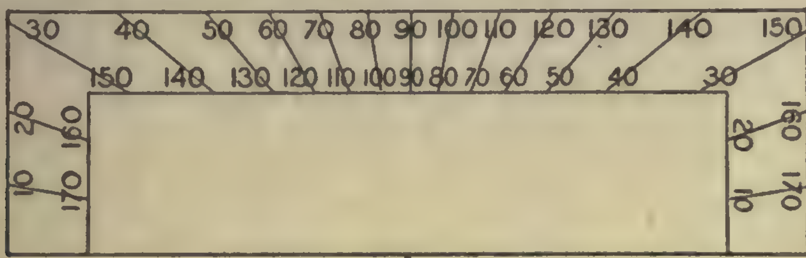
When the sum of two angles is 90°, the one is said to be the **complement** of the other.

When the sum of two angles is 180°, the one is said to be the **supplement** of the other.

A **protractor** is an instrument for constructing angles of given magnitude, and for measuring angles



that are constructed. Two forms of the protractor are here given. Its use is evident. In the first form the



graduations are at equal intervals, since, as will subsequently be shown, in a circle equal arcs subtend equal angles at the centre.

20. An angle less than a right angle, is called an **acute** angle.

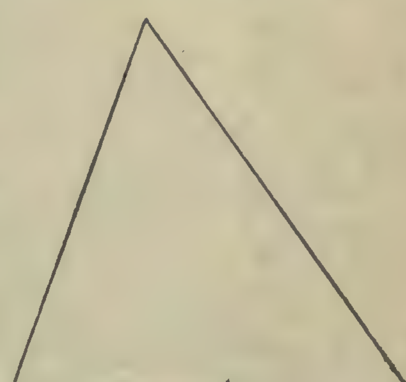
An angle greater than a right angle, is called an **obtuse** angle.

Rectilineal Figures.

21. A plane **figure** is one which is enclosed by one or more bounding lines, straight or curved, and lying in the same plane. The sum of these bounding lines is called the **perimeter** of the figure.

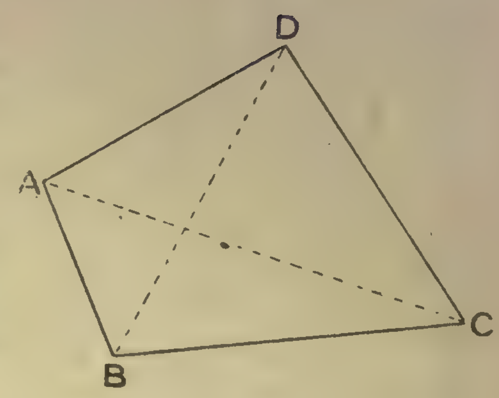
If the bounding lines are all straight lines, it is called a **plane rectilineal figure,** and these straight lines are called its sides.

22. A **triangle,** or trilateral, is a figure contained by three straight lines.



Any angular point may be called the **vertex,** and the opposite side may then be called the **base.**

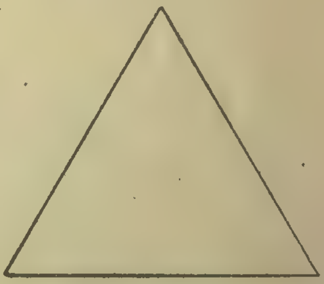
23. A **quadrilateral,** or **quadrangle,** is a plane figure contained by four straight lines, as **ABCD.**



A straight line joining opposite angular points of a quadrilateral, is called a **diagonal. AC** and **BD** are diagonals.

24. A **polygon** is a plane figure contained by more than four straight lines.

25. An **equilateral** triangle is one whose three sides are equal.



An **isosceles** triangle is one which has two equal sides.

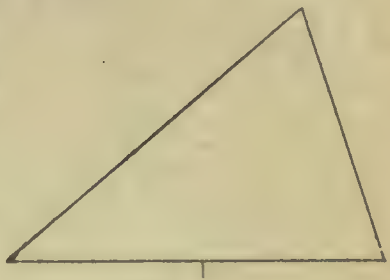
A **scalene** triangle is one which has three unequal sides.

26. A **right-angled** triangle is one which has a right angle.

The side opposite the right angle is called the **hypotenuse.**

An **acute-angled** triangle is one which has three acute angles.

It will appear later that every triangle has at least two acute angles.

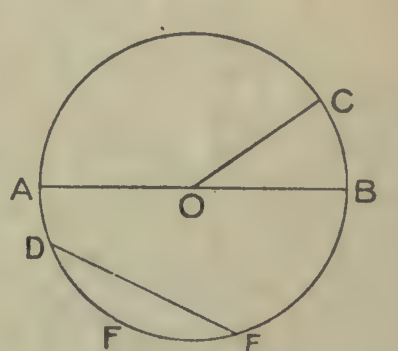


An **obtuse-angled** triangle is one which has an obtuse angle.

The Circle.

27. A **circle** is a plane figure contained by one line which is called the **circumference,** and is such that all straight lines drawn from a certain point within it, called the **centre,** to the circumference are equal to one another.

A **radius** of a circle is a straight line drawn from the centre to the circumference, as **OA,** or **OB,** or **OC.** By definition of a circle all radii of the same circle are equal to one another.



A **diameter** of a circle is a straight line drawn through the centre of the circle, and terminated both ways by the circumference, as **AOB.**

A **semicircle** is the figure contained by a diameter and the part of the circumference cut off by that diameter, as **ACBOA.**

A part of the circumference, as **DFE,** is called an **arc** of the circle.

A straight line joining two points on the circumference, is called a **chord,** as **DE.**

A **secant** is a straight line which meets the circumference of a circle in two points, cutting it in at least one.

A **segment** of a circle is the figure contained by a straight line and the part of the circumference which it cuts off, as **DEFD.**

The figure contained by two radii of a circle and the part of the circumference between their extremities, is called a **sector,** as **OBCO.**

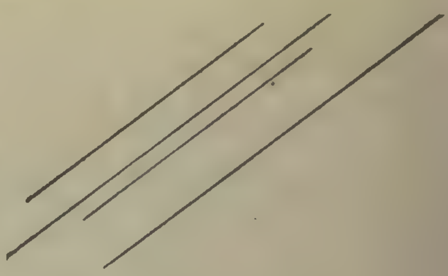
The circumference is often spoken of as the circle, and half the circumference as a semicircle.

A point is said to be on a circle when it is on the circumference.

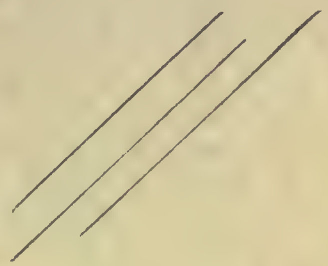
A point is said to be within a circle when it is within the circumference.

Parallel Lines.

28. **Parallel straight lines** are such as have the same direction.

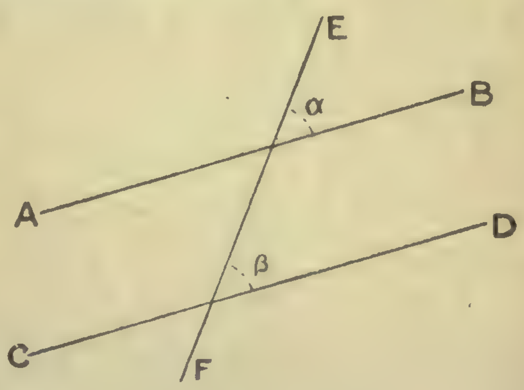


Since they have the same direction, **they can never meet on being produced ever so far either way.** For if they met they would intersect, and at the point of intersection have different directions; and therefore throughout their entire lengths they would have different directions, since each straight line maintains the same direction throughout its entire length. Hence if they met, they could not be parallel.



If two straight lines be parallel to the same straight line, they both have the same direction as this line, and therefore the same direction as one another; that is, **they are parallel to one another.**

29. Since parallel lines have the same direction, they each deviate by the same amount from any other direction; that is, they **make the same angle with any other line which intersects them.**



Thus **AB** and **CD** being parallel lines, and **FE** intersecting them, the angles α and β are equal. Here α is called the **exterior** angle, and β the **interior and opposite** angle with respect to α.

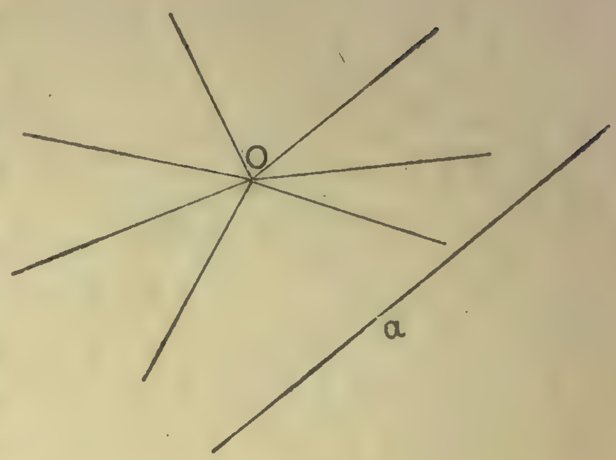
The forms **FE** and **EF** may be used to denote *opposite* directions.

Conversely, if the angles α and β be equal, the lines AB, CD deviate by the same amount from the direction FE. Hence **AB** and **CD** have the same direction, **and are therefore parallel.**

The equality of these angles, and the fact that the lines do not meet, will be taken as the characteristics of parallel lines.

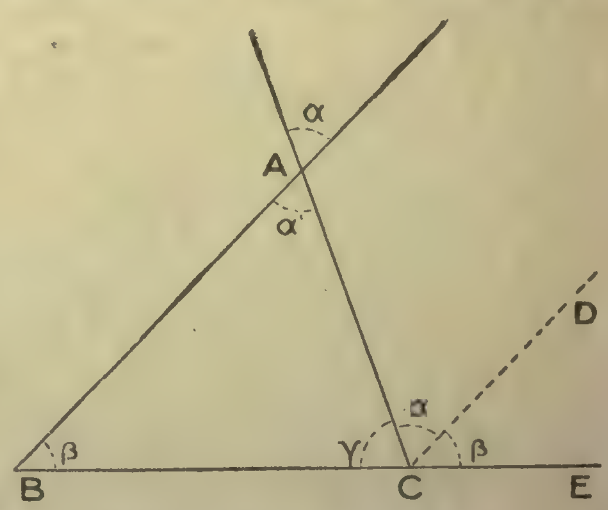
30. If a line make one complete rotation about any point **O**, in the course of this rotation it takes every

possible direction in the plane, and therefore in some position must have had the same direction as any line a.



We may express this by saying that through any point in the plane a line passes, or may be conceived as passing, parallel to any other line in the plane.

31. Let **ABC** be a triangle, and suppose (§ 30) **CD** to be the direction through **C** parallel to **BA**. Then, from the characteristic of parallel lines (§ 29), the angles β, β are equal, and also the angles α, α. But the vertically opposite angles α, α' are equal (§ 13). Hence the sum of the three angles α', β, γ of the triangle is equal to the sum of the angles at **C**, *i.e.* (§ 18), is equal to two right angles. That is, **the three angles of any triangle are together equal to two right angles.**



Also, since β at **C** is equal to β at **B**, and α at **C** to α' at **A**, therefore the angle **ACE** is equal to the sum of the angles **CAB**, **ABC**. That is, **the exterior angle of any triangle is equal to the sum of the two interior and opposite angles, and,** therefore, **is greater than either of them.**

Also, since the three angles of any triangle are together equal to two right angles, therefore **any two angles of a triangle are together less than two right angles.**

Quadrilaterals involving Parallelism.

32. A **parallelogram** is a quadrilateral whose opposite sides are parallel.

A **rectangle** is a parallelogram which has one of its angles a right angle.

It will afterwards appear that all the angles of a rectangle are right angles.

A **rhombus** is a quadrilateral all of whose sides are equal.

A **square** is a quadrilateral all of whose sides are equal, and one of whose angles is a right angle.

It will afterwards appear that all the angles of a square are right angles.

A **trapezium** is a quadrilateral which has two of its sides parallel.

The Postulates.

33. It is assumed that the following elementary constructions, called **Postulates,** are possible:

(1) *A straight line may be drawn from any one point to any other point.*

(2) *A finite (or terminated) straight line may be produced to any length in that straight line.*

(3) *A circle may be described with any given point as centre, and with any given length as radius.*

These constructions are assumed to be possible with the help of a straight ruler and a pair of compasses.

It will be noted that in the third postulate it is assumed that the compasses may be used for transferring a distance from one position to another. This assumption seems fair, since in such transference the distance between the points of the compasses is even less likely to be interfered with than in describing a circle.

Even such simple instruments as the set-square, parallel rulers, etc., are not supposed to be used. The constructions we make with them can also be made with the straight ruler and compasses, though of course not so rapidly. It is in keeping with the spirit of deductive geometry to limit, as much as possible, both the number of instruments used for construction, and also the self-evident principles on which our reasoning is based.

Axioms.

34. The following elementary truths are taken for granted as requiring no proof, or as being *self-evident.* They form the basis of our subsequent reasoning. Some of them we have already employed.

(1) *Things which are equal to the same thing are equal to one another.*

(2) *If equals be added to equals, the wholes are equal.*

(3) *If equals be taken from equals, the remainders are equal.*

(4) *If equals be added to unequals, the wholes are unequal.*

(5) *If equals be taken from unequals, the remainders are unequal.*

(6) *Things which are double of the same thing, or of equal things, are equal to one another.*

This is really a consequence of axioms 1 and 2.

We may extend this axiom, and say that things which are the same multiples of the same thing, or of equal things, are equal to one another.

(7) *Things which are halves of the same thing, or of equal things, are equal to one another.*

We may extend this axiom, and say that things which are the same sub-multiples of the same thing, or of equal things, are equal to one another.

(8) *The whole is greater than its part.*

(9) *Any figure, or diagram, can be transferred from one position to another without change of shape or size.*

Magnitudes which can be made to coincide are equal.

The placing of one geometrical magnitude upon another is called the method of **superposition,** and the one magnitude is said to be **applied** to the other.

(10) *Two straight lines cannot enclose a space.*

Already (§ 8) this axiom has been stated in other forms.

(11) *If a straight line fall on two parallel lines, it makes the exterior angle equal to the interior and opposite angle.*

This axiom has already been considered (§ 29).

That all right angles are equal is sometimes given as an axiom. This, however, has been proved (§ 17). It is an immediate result of our conception of a straight line, of the definition of a right angle, and of axiom 7.

If we wished we could add to the preceding list of self-evident truths. Thus, if **A** is equal to **B**, and **B** greater than **C**, it is evident that **A** is greater than **C**. Again, if **A** is greater than **B**, and **C** greater than **D**, it is evident that **A** together with **C** is greater than **B** together with **D**, all magnitudes being of the same kind. To these, others may be added. Axiom 4 is likely to be used in the form,—If **A** be greater than **B**, and **C** equal to **D**, then **A** together with **C** is greater than **B** together with **D**. A similar modification holds for axiom 5.

Propositions.

35. A **Proposition** in geometry is a separate discussion, and is either a **Problem** or a **Theorem.**

In a **Problem** some geometrical construction is made; in a **Theorem** some geometrical truth is established.

In a proposition we usually have:

(1) The **General Enunciation,** in which the geometrical construction to be made, or the geometrical truth to be established, is stated in general terms.

(2) The **Particular Enunciation,** in which the general enunciation is applied to a particular figure.

(3) The **Construction,** which shows what lines are to be drawn.

(4) The **Demonstration,** or **Proof,** which shows that the problem has been solved, or that the theorem is true.

A **Corollary** to a proposition is a statement of a fact which follows immediately from the proposition.

In the enunciation of a theorem, the **Hypothesis** is that which is assumed to be true, and the **Conclusion** is that which has to be proved.

One theorem is said to be the **Converse** of another when the hypothesis of the former becomes the conclusion of the latter.

SYMBOLS AND ABBREVIATIONS

The following may be used in writing out the propositions:—

=	*for*	is equal to, are equal to, equal to, equal.
∴	..	therefore.
∠	..	angle.
rt. ∠	..	right angle.
⊥ *or* perp.		perpendicular.
△	..	triangle.
○	..	circle.
○*ce.*	..	circumference.
>	..	is greater than, are greater than, greater than, be greater than.
<	..	is less than, are less than, less than, be less than.
‖	..	parallel.
‖*m*	..	parallelogram.

In.	*for*	Introduction.
prop.	..	proposition.
cor.	..	corollary.
hyp.	..	hypothesis.
cr.	..	centre.
const.	..	construction.
gr.	..	greater.
opp.	..	opposite.
int.	..	interior.
ext.	..	exterior.
alt.	..	alternate.
adj.	..	adjacent.
sq.	..	square.
rect.	..	rectangle.
quadl.	..	quadrilateral.
rad.	..	radius.
isos.	..	isosceles.
=lat.	..	equilateral.
=∠*r*	..	equiangular.
st. line	..	straight line.
pt.	..	point.

BOOK I.

TRIANGLES.

Congruent Triangles with Subsidiary Constructions and Theorems.

Bisections.

Perpendiculars.

Loci.

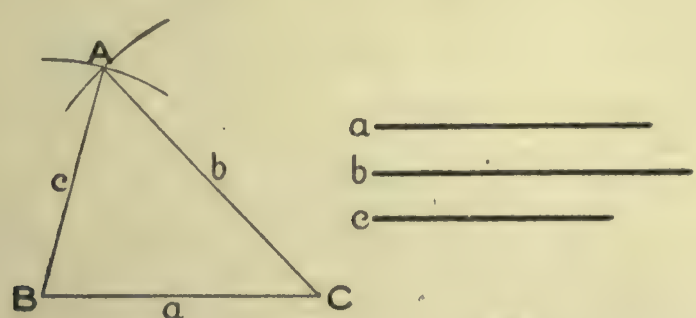
Angles of a Polygon.

Certain Geometrical Inequalities.

Congruent Triangles with Subsidiary Constructions and Theorems.

PROPOSITION I. PROBLEM.

To construct a triangle of which the sides shall be equal to three given straight lines, any two whatever of which are together greater than the third.



Let a, b, c be the three given st. lines, any two being greater than the third.

It is required to construct a △ *whose sides shall be* = to a, b, c.

Take any st. line **BC** $= a$.

With cr. **B**, and radius $= c$, describe an arc of a circle.

With cr. **C**, and radius $= b$, describe an arc of a circle, cutting the former arc at **A**.

Join **AB**, **AC**.

Then **ABC** is the required △.

For by construction the three sides, **BC**, **CA**, **AB**, are = to the three given st. lines a, b, c, respectively.

Since a straight line is the shortest distance between its ends (In., § 6), any two sides of a triangle are together greater than the third side. Hence the condition "any two whatever of which are together greater than the third" becomes necessary.

If one of the given lines be greater than the sum of the other two, it will be found that the circles will not intersect, one lying either wholly within, or wholly without, the other.

The construction of the Proposition, of course, includes the construction of equilateral and isosceles triangles as particular cases.

If the circles be more fully drawn, they will intersect also below **BC.** We thus get a second triangle, on the other side of **BC,** whose sides are of the required lengths.

Exercises.

1. Describe two equilateral triangles with a given straight line AB as common side.

2. On a given straight line AB as base, describe an isosceles triangle with sides twice AB.

3. Describe a triangle with sides 2, 3 and 4 inches.

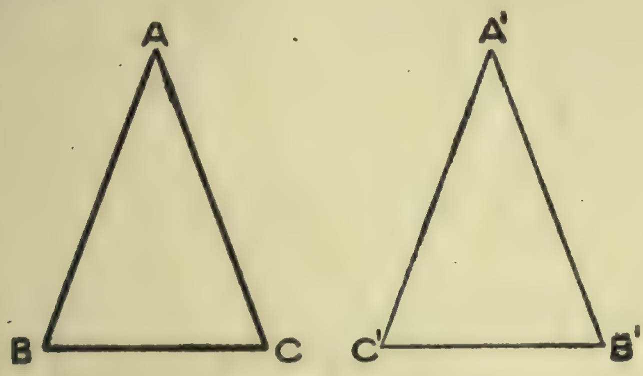
4. Describe a triangle with sides 60, 70 and 100 millimetres.

5. In the Proposition b and c being unequal, describe on the same side of BC a second triangle whose sides are equal to a, b and c.

6. In the preceding question show that the vertex of the second triangle cannot coincide with the vertex of the first.

PROPOSITION II. THEOREM.

The angles at the base of an isosceles triangle are equal.



Let ABC be an isosceles $\triangle$, having side AB = side AC.

Then $\angle$ABC = $\angle$ACB.

Reverse the $\triangle$ ABC, leaving its trace behind, so that it takes the position A′C′B′, C′ being the new position of C, B′ the new position of B, and A′ of A.

Apply the $\triangle$A′C′B′ to the $\triangle$ABC, so that A′ falls on A, and A′C′ on AB.

Then A′B′ will fall upon AC, because $\angle$C′A′B′ is $\angle$BAC in another position.

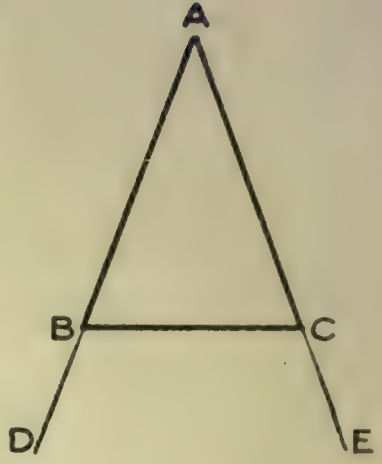
Also C′ will fall on B, because A′C′ = AC = AB.

And B′ will fall on C, because A′B′ = AB = AC.

$\therefore$ C′B′ coincides with BC.

Hence $\angle$A′C′B′, which is $\angle$ACB, coincides with and is equal to $\angle$ABC.

Cor. 1. If the equal sides **AB, AC** be produced to **D** and **E**, the $\angle$s **DBC, ECB**, on the other side of the base, are also equal.



For

$$\angle ABC + \angle DBC = 180°$$
$$= \angle ACB + \angle ECB. \quad \text{(In., § 18.)}$$
$$\text{And } \angle ABC = \angle ACB.$$
$$\therefore \angle DBC = \angle ECB.$$

Cor. 2. All the angles of an equilateral triangle are equal to one another. Each of them is $\therefore$ 60°. (In., § 31.)

Exercises.

1. If two isosceles triangles are on the same base and on the same side of it, one triangle is entirely within the other.

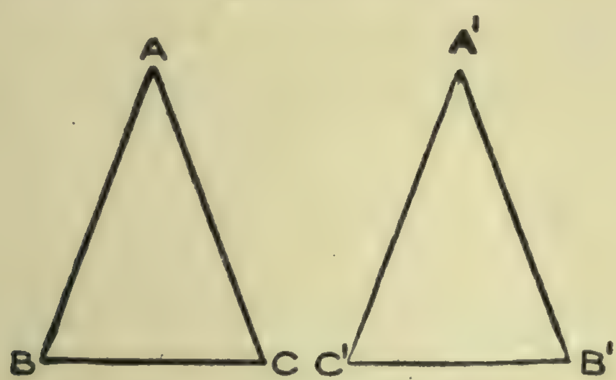
2. Prove that the opposite angles of a rhombus are equal to one another.

3. If two angles of a triangle are unequal, the sides opposite to them are also unequal.

4. ABC is an isosceles triangle having AB equal to AC. BA is produced to D. Prove that angle DCB is greater than DBC.

PROPOSITION III. THEOREM.

If two angles of a triangle be equal, the sides opposite to them are also equal.



Let $\triangle ABC$ have $\angle ABC = \angle ACB$.

Then side AB = *side* AC.

Reverse $\triangle$ ABC, leaving its trace behind, so that it takes the position A′C′B′, C′ being the new position of C, B′ the new position of B, and A′ of A.

Apply the $\triangle A'C'B'$ to the $\triangle ABC$, so that C′ rests on B, and C′B′ on BC.

Then B′ coincides with C, because $C'B' = BC$.

Also since $\angle B'C'A' = \angle BCA = \angle CBA$; $\therefore$ C′A′ falls on BA.

And since $\angle C'B'A' = \angle CBA = \angle BCA$; $\therefore$ B′A′ falls on CA.

Hence A′ lies on both the lines BA and CA. or on th

$\therefore$ it coincides with A.

And C′A′, *i.e.*, CA, is equal to BA.

Cor. If a triangle be equiangular it is also equilateral.

NOTE: Proposition III. is the *converse* of Proposition II. Possibly it might have been more logically introduced after we have seen how to construct an angle equal to another. It is placed here that these converse propositions may be associated. No immediate use is made of it.

Exercises.

1. If two sides of a triangle are unequal, the angles opposite to them are also unequal.

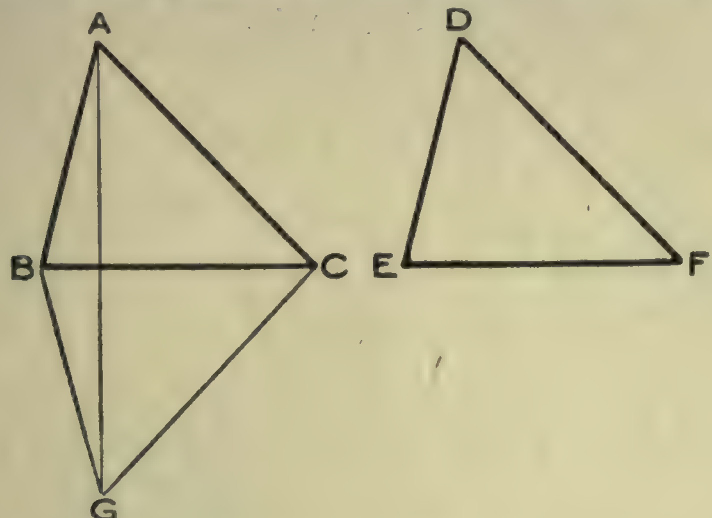
2. If when two sides of a triangle are produced the exterior angles are equal, show that the triangle is isosceles.

3. ABC is a triangle having the angle ABC double the angle BAC. If BD bisect the angle ABC, and meet AC in D, show that DA = DB.

4. If in the triangle ABC, the angles B and C be each double the angle A, and BD bisect the angle B, what three lines in the figure are equal to one another?

PROPOSITION IV. THEOREM.

If the three sides of one triangle be respectively equal to the three sides of another triangle, the triangles are equal in every respect.



Let **ABC**, **DEF** be the △s, having **AB** = **DE**, **BC** = **EF**, **CA** = **FD**.

Then the △s are equal in all respects.

Apply the △**DEF**, so that **E** rests on **B**, and **EF** on **BC**.

Then **F** falls on **C**, since **EF** = **BC**.

Let **DEF** take the position **GBC**. Join **AG**.

Because **BG** = **BA**; ∴ ∠**BGA** = ∠**BAG**. (Prop. 2.)

Because **CG** = **CA**; ∴ ∠**CGA** = ∠**CAG**.

Hence the whole ∠**BGC** = the whole ∠**BAC**.

That is, ∠**EDF** = ∠**BAC**.

Evidently, in like manner, it follows that ∠**DEF** = ∠**ABC**, and ∠**DFE** = ∠**ACB**.

That is, in the two △s, the ∠s which are opposite equal sides are equal.

The ∠**GBC** being = ∠**ABC**, and ∠**GCB** = ∠**ACB**, suppose △**GBC** to rotate about **BC** into coincidence with **ABC**.

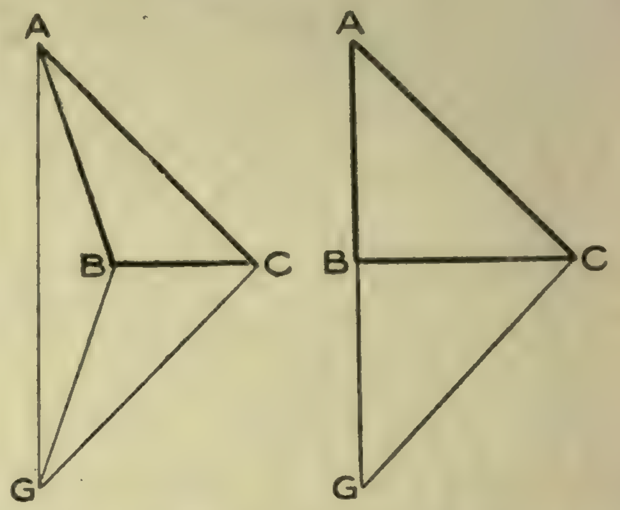
BG will coincide with **BA**, because ∠**GBC** = ∠**ABC**.

Also, **CG** will coincide with **CA**, because ∠**GCB** = ∠**ACB**.

Hence **G**, lying on both **BA** and **CA**, will coincide with

A; and the $\triangle$ **GBC** coincides with and is equal to the $\triangle$ **ABC**.

That is, $\triangle$ **DEF** = $\triangle$ **ABC** in area.



NOTE 1. The pupil may be left to modify the preceding proof to meet the case where **AG** falls without the triangles, or passes through **B** or **C**.

NOTE 2. Other ways of stating the preceding proposition are to say that two such triangles are the same triangle in different positions; or that if the sides of a triangle are fixed, the angles are fixed, and the area is fixed.

Exercises.

1. ABC, DBC are two isosceles triangles on the same base BC, but upon opposite sides of it. Show that AD bisects the angles BAC, BDC.

2. State the preceding proposition when the isosceles triangles are on the same side of BC.

3. On a given line BD as diagonal, construct a quadrilateral ABCD, such that AD = BC, and AB = DC. Examine what angles in the figure are equal to one another.

4. Equilateral triangles on equal bases are equal in all respects.

5. Two circles whose centres are A and B intersect in C and D. Show that the triangles CAB, DAB are equal in all respects.

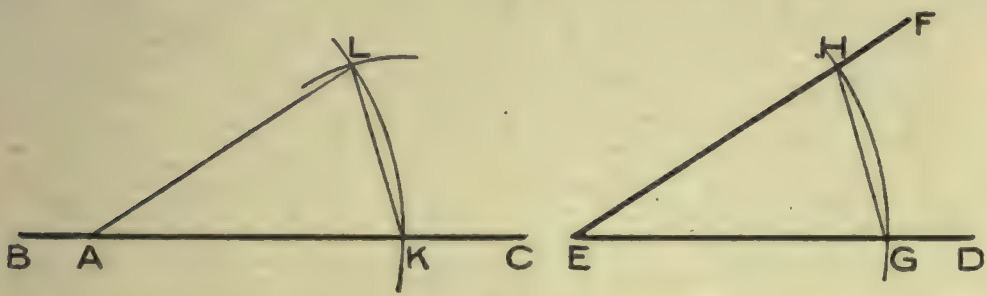
6. On a diagonal AC, 3 inches in length, construct a rhombus ABCD with sides $1\frac{3}{4}$ inches in length. Show that AC bisects the angles at A and C.

7. Prove that the opposite angles of a rhombus are equal.

8. ACB, ADB are two triangles on the same side of AB, with AC = BD and AD = BC. If AD, BC meet in O, prove that the triangles OAB and OCD are isosceles.

PROPOSITION V. PROBLEM.

At a given point in a given straight line to construct an angle equal to a given angle.



Let **A** be the given pt. in the given straight line **BC**, and **DEF** the given $\angle$.

It is required to construct at **A** *an* $\angle$ *equal to* **DEF**, *and such that* **AC** *shall be one of its bounding lines.*

With **E** as centre, describe a circle cutting **ED** in **G** and **EF** in **H**. Join **GH**.

With **A** as centre and radius = **EG**, describe a circle cutting **AC** in **K**.

With **K** as centre and radius = **GH**, describe a circle cutting the preceding in **L**.

Join **AL**, **LK**.

Then in $\triangle$s **AKL**, **EGH**,

AK = **EG**,

KL = **GH**,

LA = **HE**;

$\therefore$ $\triangle$s are equal in all respects (Prop. 4);

and $\angle$**KAL** = $\angle$**GEH** = $\angle$**DEF**.

Exercises.

1. With a protractor construct an angle of 59°, and by the method of the proposition construct an angle equal to it. With the protractor test the accuracy of the construction.

2. In the side AB, or in AB produced, of a triangle ABC, find a point equidistant from B and C.

3. On a given line as base, construct an isosceles triangle with each of the angles at the base equal to a given angle.

4. Construct a triangle, having given the base, one of the angles at the base, and the sum of the sides.

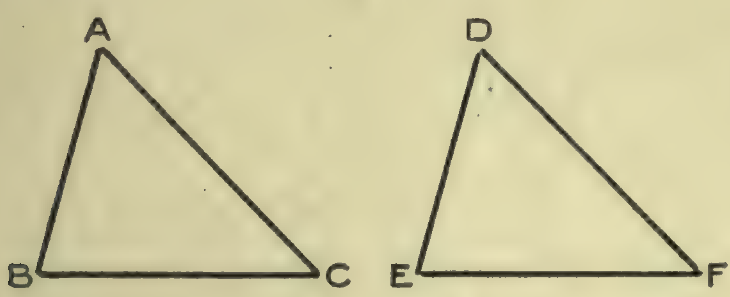
5. Construct a triangle, having given the base, one of the angles at the base, and the difference of the sides.

6. A is a point without a line BC of given length. Find a point P in BC, such that $AP + PB = CB$.

7. A is a given point, and B is a given point in a given straight line. Find a point P in the given line, such that the sum of AP and PB may be equal to a given length.

PROPOSITION VI. THEOREM.

If two triangles have two sides of one equal respectively to two sides of the other, and the included angles equal, the triangles are equal in all respects.



Let ABC, DEF be two $\triangle$s, such that AB = DE, AC = DF, and $\angle$ BAC = $\angle$ EDF.

Then the $\triangle$s are equal in all respects.

Apply the $\triangle$DEF to the $\triangle$ABC, so that D rests on A, and DE on AB.

Then DF falls on AC, because $\angle$ EDF = $\angle$ BAC.

And DE falling on AB, E must coincide with B, because DE = AB.

Also DF falling on AC, F must coincide with C, because DF = AC.

$\therefore$ $\triangle$DEF coincides with $\triangle$ABC, and is equal to it in all respects.

That is, BC = EF, $\angle$ ABC = $\angle$ DEF, $\angle$ ACB = $\angle$ DFE, and $\triangle$s are equal in area.

NOTE. Two such triangles are indeed the same triangle in different positions.

Another way of stating the proposition is to say that if two sides and the included angle of a triangle are fixed, the remaining side and angles are fixed, and the area is fixed.

Exercises.

1. Two straight lines AB, CD bisect one another at E. Prove that the triangles AEC, BED are equal in all respects. Also the triangles BEC, AED.

2. With the vertex A of an isosceles triangle ABC as centre, a circle is described which cuts the equal sides AB, AC in D and E respectively. Show that the triangles ACD, ABE are equal in all respects.

3. The sides AB, AD of a quadrilateral ABCD are equal, and the diagonal AC bisects the angle BAD; prove that the sides CB, CD are equal, and that the diagonal AC bisects the angle BCD.

4. Two quadrilaterals ABCD, EFGH have $AB = EF$, $BC = FG$, $CD = GH$, $\angle ABC = \angle EFG$, $\angle BCD = \angle FGH$. Prove that the quadrilaterals are equal in all respects.

5. Two points in the base of an isosceles triangle are equidistant from the ends of the base. Show that they are also equidistant from the vertex.

6. Show that the diagonals of a rhombus bisect one another at right angles.

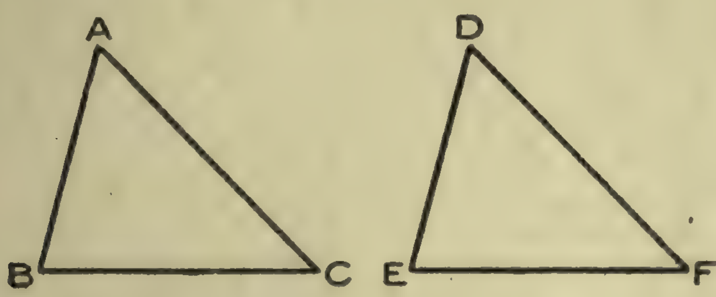
7. On opposite sides of AB equal angles BAC, BAD are constructed, and AC is taken equal to AD. Show that AB bisects CD at right angles.

8. The equal sides AB, AC of an isosceles triangle are produced, and E and F are taken in the productions, so that $AE = AF$. BF and CE are joined. Show that $BF = CE$.

9. In the preceding question, if BF and CE intersect in O, show that AO bisects the angle BAC.

PROPOSITION VII. THEOREM.

If two triangles have two angles of one equal to two angles of the other, each to each, and therefore (In., § 31) **the third angles in each equal, and a side of the first equal to the corresponding side of the other, the triangles are equal in all respects.**



Let **ABC**, **DEF** be two $\triangle$s in which $\angle$**BAC** = $\angle$**EDF**, and $\angle$**ABC** = $\angle$**DEF**,

and consequently $\angle$**ACB** = $\angle$**DFE** (In., § 31);

and also let **BC** = **EF**.

Then the $\triangle$s *are equal in all respects.*

Apply the $\triangle$**DEF** to the $\triangle$**ABC**, so that **E** falls on **B**, and **EF** on **BC**.

Then **F** must coincide with **C**, because **EF** = **BC**.

Also because $\angle$**FED** = $\angle$**CBA**, the side **ED** must coincide with **BA**.

And because $\angle$**EFD** = $\angle$**BCA**, the side **FD** must coincide with **CA**.

Hence the point **D**, which falls on both **BA** and **CA**, must coincide with **A**, where **BA**, **CA** intersect.

Therefore the $\triangle$**DEF** coincides with the $\triangle$**ABC**, and is equal to it in all respects.

So that **AB** = **DE**, **AC** = **DF**, and the $\triangle$s are equal in area.

NOTE. Two such triangles are the same triangle in different positions.

Another way of stating the proposition is to say that if two angles of a triangle are fixed, and a side also fixed (whether it be adjacent to both given angles or adjacent to one and opposite the other), then the remaining angle and sides are fixed, and the area is fixed.

Propositions IV., VI., and VII. are very important. Their real significance may be expressed thus:

A triangle is fixed and determinate if

(1) **Its three sides are given.**

(2) **Two sides and the included angle are given.**

(3) **One side and two angles are given.**

Two triangles which are equal in all respects, so that the one may be made to coincide with the other, are said to be **congruent.** Two triangles which have the same area may be said to be *equal,* though differing in shape.

Exercises.

1. On a given line as diagonal, construct a quadrilateral, so that this diagonal shall bisect the angles through which it passes; and show that the other diagonal is bisected at right angles by this.

2. From the sides AB, BC, CA of an equilateral triangle ABC, equal lengths AF, BD, CE are cut. BE and CF intersect in G, CF and AD in H, AD and BE in K. Show that the triangles CDH, AEK, BFG are equal in all respects.

Show also that the triangle GHK is equilateral.

PROPOSITION VIII. THEOREM.

If two sides of one triangle be respectively equal to two sides of another triangle, and the angles opposite to one pair of equal sides be equal, then the angles opposite to the other pair of equal sides are either equal or supplementary; and, if equal, then the triangles are equal in all respects.

FIG. 1.

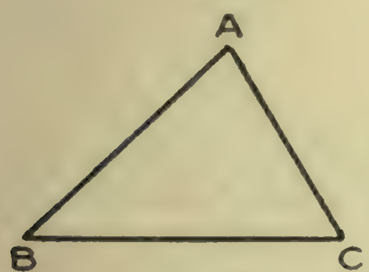
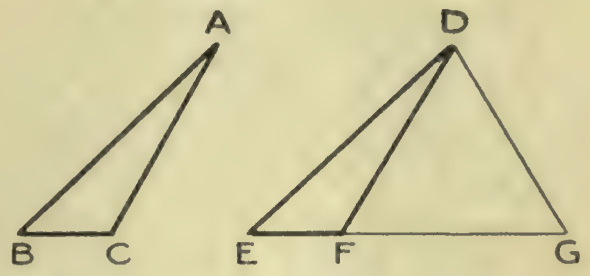


FIG. 2.



In the △s **ABC, DEF**, let **AB** = **DE**, **AC** = **DF**, and ∠**ABC** = ∠**DEF**.

Then the ∠s **ACB, DFE** *are either equal or supplementary.*

If the ∠**ACB** (Fig. 2) and the ∠**DFE** be equal, the third ∠s **BAC, EDF** are equal; and the sides **AB, DE** being equal, the △s **ABC, DEF** are equal in all respects.

If ∠**ACB** (Fig. 1) be not = ∠**DFE**, make ∠**EDG** = ∠**BAC**.

Then in △s **ABC, DEG**,

∠**ABC** = ∠**DEG**,

∠**BAC** = ∠**EDG**,

side **AB** = side **DE**;

∴ these △s are equal in all respects. (Prop. 7.)

Hence **DG** = **AC** = **DF**;

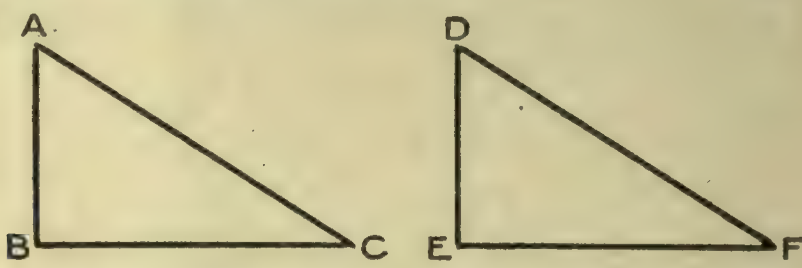
∴ ∠**DFG** = ∠**DGF** = ∠**ACB**.

But ∠**DFG** is supplementary to ∠**DFE**;

∴ ∠**ACB** is supplementary to ∠**DFE**.

Cor. The following corollary of Prop. VIII. is important:

If two right-angled triangles have their hypotenuses equal, and one side of the one equal to one side of the other, the triangles are equal in all respects.



Let △s **ABC, DEF** have right ∠s at **B** and **E**; also **AB** = **DE** and **AC** = **DF**: then △s are equal in all respects.

For by Prop. VIII. the ∠s at **C** and **F** are either equal or supplementary.

They cannot be supplementary, since they are both acute (In., § 31), the ∠s at **B** and **E** being right ∠s.

Hence the ∠s at **C** and **F** are equal;

∴ ∠s at **A** and **D** are equal (In., § 31);

and △s are equal in all respects (Prop. 6).

NOTE. In referring to the various cases in which two triangles are congruent, Prop. VIII. is often spoken of as the **ambiguous case**.

Exercises.

1. In the triangles ABC, DEF, the angles ABC, DEF being equal, and the angles ACB, DFE being supplemental, if AB be equal to DE, show that AC is equal to DF.

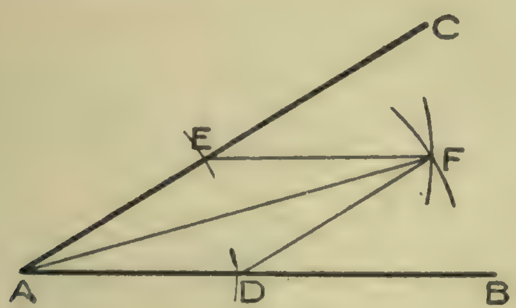
2. In the preceding, the angles ABC, DEF being equal, and the angles ACB, DFE supplemental, if AC be equal to DF, show that AB is equal to DE.

3. **Give an alternative proof of the proposition stated in the Cor. to Prop. viii.,** making AB coincide with DE, and placing CB in same st. line with EF.

Bisections.

PROPOSITION IX. PROBLEM.

To bisect a given angle.



Let **BAC** be the given $\angle$.

It is required to bisect it.

With centre **A** and any radius, describe an arc of a circle cutting **AB**, **AC** in **D** and **E** respectively.

With centres **D** and **E** and any equal radii, describe arcs of circles intersecting in **F**.

Join **AF**. It bisects the $\angle$**BAC**.

Join **FD**, **FE**.

In $\triangle$s **DAF**, **EAF**,

AD = **AE**,

AF is common to $\triangle$s,

DF = **EF**.

Hence $\angle$**DAF** = $\angle$**EAF**; (Prop. 4.)

and **AF** bisects $\angle$**BAC**.

Exercises.

1. Construct angles of 45°, 54°, 60°, 138°, and bisect them by the method of the proposition.

2. For the same angle BAC, take the radius AD of various lengths, and also the radius DF(=EF) of various lengths, and show, by accurate construction, that the same bisecting line is always obtained.

3. Show that any point in the bisector of the vertical angle of an isosceles triangle is equidistant from the extremities of the base.

4. ABC is an isosceles triangle, and the equal angles at B and C are bisected by lines which meet the opposite sides in E and D. Show that BE=CD.

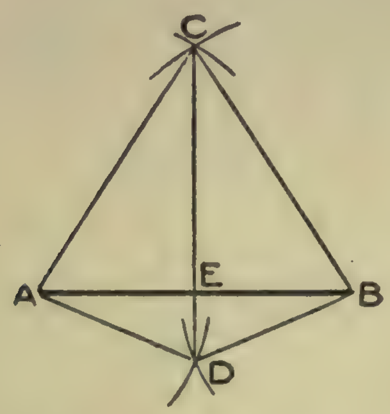
5. ABC is an isosceles triangle, and the equal angles at B and C are bisected by lines which meet in O. Show that BO=CO. Also show that AO bisects the angle at A.

6. Show that, in an isosceles triangle, the bisector of the vertical angle bisects the base at right angles.

7. In the Proposition show that any point in AF, or AF produced either way, is equally distant from D and E.

PROPOSITION X. PROBLEM.

To bisect a given straight line.



Let **AB** be the given st. line.

It is required to bisect it.

With centres **A** and **B**, and equal radii, describe arcs of circles intersecting at **C**.

With centres **A** and **B**, and equal radii, describe arcs of circles intersecting at **D**.

Join **CD**, cutting **AB** in **E**.

AB is bisected at **E**.

Join **AC**, **BC**, **AD**, **BD**.

In $\triangle$s **ACD**, **BCD**,

AC = **BC**,

CD is common to $\triangle$s.

AD = **BD**;

$\therefore$ $\angle$**ACD** = $\angle$**BCD**. (Prop. 4.)

Again, in $\triangle$s **ACE**, **BCE**,

AC = **BC**,

CE is common to $\triangle$s,

$\angle$**ACE** = $\angle$**BCE**;

$\therefore$ **AE** = **BE**; (Prop. 6.)

and **AB** is bisected at **E**.

NOTE. The radii of all the circles might have been the same, and the trouble of readjusting the compasses would have been saved. The method of the proposition, however, has been followed in order to indicate what was essential,—the equality of the radii in pairs.

Evidently the radii must be greater than half the length of **AB**, that the circles may intersect.

Exercises.

1. If, in the Proposition, the radii BC, AD be equal, and also the radii AC, BD equal, show that CD still bisects AB, and that it is bisected by AB.

2. Prove that the straight lines which join the middle points of the equal sides of an isosceles triangle to the ends of the base, are equal.

3. A straight line AB is bisected at C, and on the same side of the line triangles ADC, BEC are described, having AD, DC respectively equal to BE, EC. Prove that AE is equal to BD.

4. The bisector of the vertical angle of a triangle also bisects the base. Show that the triangle is isosceles. Could the angles at the base be supplemental?

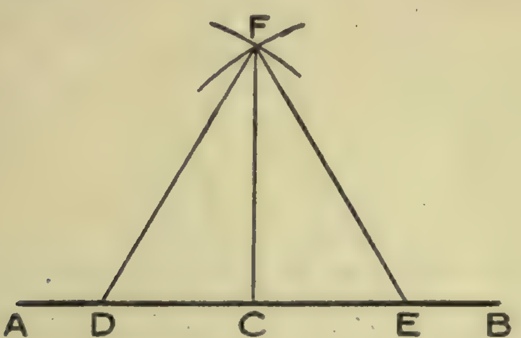
5. Show how to bisect AB by describing only two circles.

6. In the Proposition show that if a point be equally distant from the points A and B, it must lie in the line CD.

Perpendiculars.

PROPOSITION XI. PROBLEM.

To draw a perpendicular to a given straight line from a given point in it.



Let AB be the given st. line, and C the given pt. in it.

It is required to draw from C *a st. line* ⊥r to AB.

Take $CD = CE$.

With centres D and E, and equal radii, describe arcs of circles intersecting at F.

Join CF.

CF is ⊥r to AB.

Join DF and EF.

In △s DCF, ECF,

$DC = EC$,

CF is common to △s,

$DF = EF$;

∴ $\angle DCF = \angle ECF$; (Prop. 4.)

But they are adjacent ∠s;

∴ each is a right ∠; (In., § 15.)

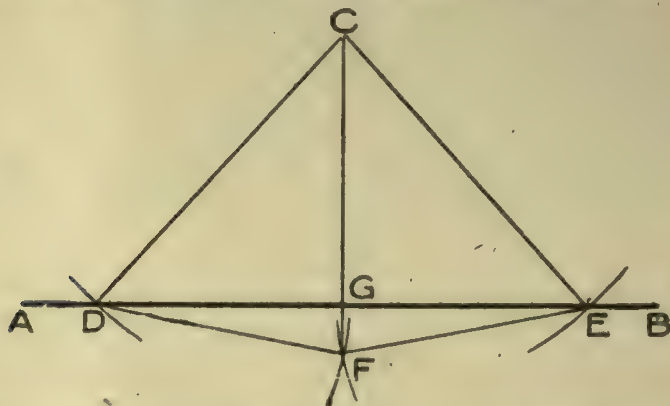
and CF is ⊥r to AB.

NOTE: It will be observed that Prop. xi. is a particular case of Prop. ix., the angle to be bisected in Prop. xi. being a *straight* angle. The constructions and proofs are practically the same in both propositions.

PROPOSITION XII. PROBLEM.

Jm

To draw a perpendicular to a given straight line of unlimited length from a given point not on the line.



Let **AB** be the given st. line, and **C** the given pt.

It is required to draw from **C** *a st. line* ⊥r *to* **AB**.

With centre **C**, describe an arc of a circle to cut **AB** in **D** and **E**.

With **D** and **E** as centres, and equal radii, describe arcs of circles intersecting at **F**.

Join **CF**, cutting **AB** in **G**.

Then **CG** is ⊥r to **AB**.

In △s **DCF**, **ECF**,

DC = **EC**,

CF is common to △s,

DF = **EF**;

∴ ∠**DCF** = ∠**ECF**. (Prop. 4.)

Again, in △s **DCG**, **ECG**,

DC = **EC**,

CG is common to △s,

∠**DCG** = ∠**ECG**;

∴ △s are equal in all respects; (Prop. 6.)

and ∠**DGC** = ∠**EGC**.

But they are adjacent ∠s;

∴ each is a rt. ∠; (In., § 15.)

and **CG** is ⊥r to **AB**.

Exercises.

1. If one angle of a triangle is equal to the sum of the other two, what is the first angle?

2. Divide a right-angled triangle into two isosceles triangles.

3. From the end of a line draw a perpendicular to it without producing the line.

4. Two straight lines AB, CD intersect in O; and the angles AOC, AOD are bisected by OE, OF. Show that OE is at right angles to OF.

5. From D, which is not in either of the lines AB, AC, draw a line DEF which shall cut off equal lengths AE, AF from AB, AC.

6. Construct an isosceles triangle, having given the vertical angle and the perpendicular from the vertical angle on the base.

7. Construct an isosceles triangle, having given its perimeter and the perpendicular from the vertex on the base.

8. Show that, if perpendiculars be dropped on the arms of an angle from any point in the line bisecting the angle, these perpendiculars are equal.

9. If C be the middle point of AB, and CD be drawn perpendicular to AB, then every point in CD is equidistant from A and B.

10. Perpendiculars drawn from the ends of the base of an isosceles triangle to the sides are equal.

11. Three straight lines AB, AC, AD are drawn from A, and the angles BAC, CAD are bisected by the straight lines AE, AF. Show that if the angle EAF be a right angle, the lines AB, AD are in the same straight line.

12. If the bisector of an angle of a triangle be perpendicular to the opposite side, the triangle is isosceles.

13. A straight line ABC is drawn on a sheet of paper, which is then folded about B, so that BC falls on BA. Show that the crease in the paper is perpendicular to BA and BC.

14. In a triangle ABC, perpendiculars are dropped from B and C on the opposite sides. Show that the angles these perpendiculars make with one another are B + C and A.

15. A and B are two given points in the plane of the paper. Find a straight line such that if the paper be folded about it, A shall coincide with B.

Additional Exercises.

1. Four straight lines meet in a point in such a way that opposite angles are equal to one another. Prove that the lines are, two and two, in the same straight line.

2. If two straight lines intersect, show that the bisectors of vertically opposite angles are in the same straight line.

3. The opposite sides AB, CD of a quadrilateral ABCD are equal, and the straight lines bisecting AD, BC at right angles meet in O. Show that the triangles OAB, ODC are equal in all respects.

4. On a given straight line as diagonal, construct an equilateral four-sided figure (*i.e.*, a rhombus), the sides being of given length.

What limitation is there as to lengths of the sides?

5. Show that in the rhombus

(1) The opposite angles are equal.

(2) The diagonals bisect the angles through which they pass.

(3) The diagonals bisect one another at right angles.

6. Construct a rhombus with sides of given length, and with one of its angles given.

Show that this angle with the other angle of the rhombus make up two right angles (In., § 31).

7. In the preceding question, when the given angle of the rhombus is a right angle, show that all the angles are right angles, *i.e.*, that the figure you have constructed is a square.

8. On a given line as diagonal, construct a four-sided figure having its opposite sides equal.

What limitation is there as to lengths of sides?

9. Show that in the quadrilateral with opposite sides equal, constructed in preceding question,

(1) The opposite angles are equal.

(2) A diagonal divides the angles through which it passes into angles that are equal alternately.

(3) A diagonal divides the figure into triangles that are equal in area.

(4) The diagonals bisect one another.

10. Construct a quadrilateral with opposite sides equal and of given magnitude, and with one of its angles given.

Show that this angle with the other angle of the figure make up two right angles.

11. On the circumference of a circle whose centre is O, three points A, B, C are taken, such that the straight lines AB, BC are equal. Show that OB bisects AC at right angles; also that OB bisects the angles AOC, ABC.

12. From the ends B, C of the base of a triangle ABC, straight lines are drawn intersecting in F and meeting the opposite sides, or opposite sides produced in D and E. Show that if FB = FC, and FD = FE, the triangle is isosceles.

13. A straight line AOB, in which OA = OB, rotates about the fixed point O. Show that the perpendiculars from A and B on any line through O are always equal to one another.

14. If in a quadrilateral ABCD, the sides AB and CD be equal, show that the line joining the middle points of BC and AD is equally inclined to AB and CD. (Use preceding exercise.)

15. In a right-angled triangle the hypotenuse is double the line from its middle point to the right angle.

16. If, in a right-angled triangle, one of the acute angles be double the other, show that the hypotenuse is double the smaller side.

How many degrees are there in each angle of the figure?

17. If, in a right-angled triangle, the hypotenuse be double the smallest side, show that one of the acute angles is double the other.

18. From two given points on the same side of a given line, draw two lines which shall meet in that line and make equal angles with it.

19. Through two given points on opposite sides of a given straight line, draw two straight lines which shall meet in the given straight line, and include an angle bisected by the given straight line.

20. Prove by superposition that if all the sides of one quadrilateral be equal respectively and in order to the sides of another quadrilateral, and if also an angle in one be equal to the corresponding angle in the other, then the quadrilaterals are equal in all respects.

21. In the quadrilateral ABCD, the sides AB and AD are equal, and the angles ABC, ADC are equal; show that BC and CD are equal.

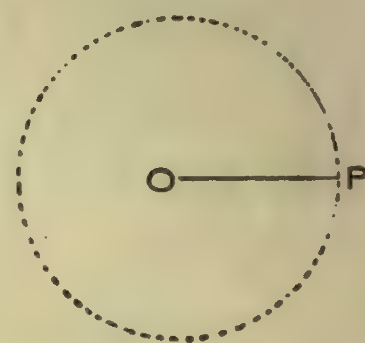
22. Construct a right-angled triangle, having given the lengths of the hypotenuse and of one side.

23. If the sides AB, AC of a triangle ABC be produced to D and E, and if the bisectors of the angles BCE, CBD meet in O, show that the perpendiculars from O on BD, BC, CE are all equal.

Loci.

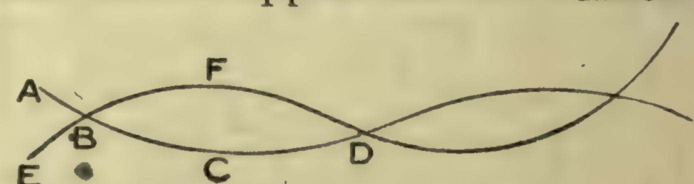
Locus of a point.—*The locus of a point is the path traced out by the point when it moves in accordance with some fixed law.*

Thus if a point **P** move so that its distance from a fixed point **O** is always the same (**OP**), the locus of **P** is the circumference of the circle whose centre is **O** and radius **OP**.

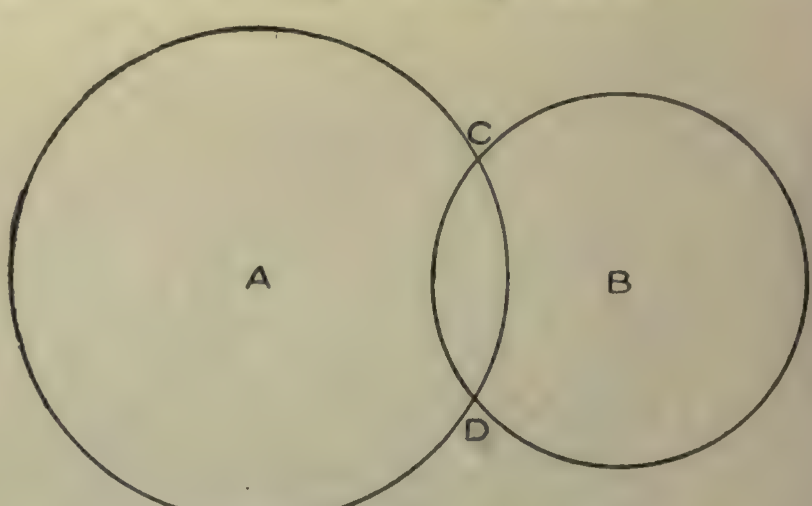


The *law* in this case is the constancy of the distance of **P** from the fixed point **O**.

One of the most-valuable applications of the idea of locus may be illustrated thus: Suppose the curve **ABCD..** to be the locus of a point which satisfies one set of conditions (*i.e.*, follows one law); and the curve **EBFD..** to be the locus of a point which satisfies another set of conditions (*i.e.*, follows another law); then the points of intersection of the curves, **B, D, . . ,** evidently are points which satisfy both sets of conditions, *i.e.*, obey both laws.



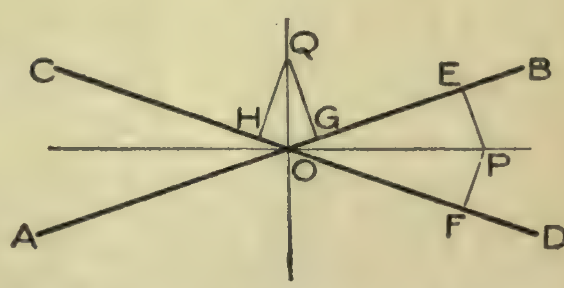
Thus, if one of these circles be the locus of a pt. which is at a distance of 20 millimetres from **A**, and the other



circle be the locus of a point which is at a distance of 15 millimetres from B, then C and D are points which fulfil both conditions, *i.e.*, are 20 millimetres from A and 15 millimetres from B.

The student will at once see that as far back as Prop. I. we were covertly using the notion of locus. We wanted a point that was at a distance c from B, and a distance b from C. Accordingly we constructed part of the locus of points at a distance c from B, and part of the locus of points at a distance b from C. The intersection of these loci gave the point A sought.

Ex. 1. *To find the locus of a point which moves so that its distances from two given straight lines* AB, CD *are equal to one another.*

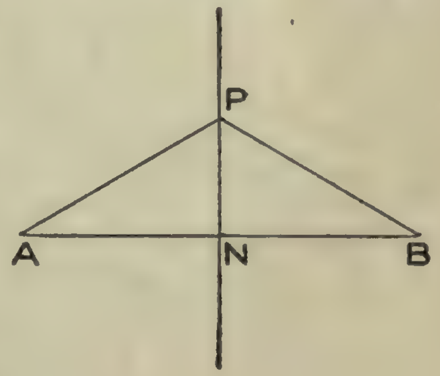


Evidently the locus sought is one or other of the lines OP, OQ which bisect the angles between AB, CD.

It is left to the student to show that, whatever be the position of P on OP, PE = PF; and that, whatever be the position of Q on OQ, QG = QH.

Also that no point not on OP or OQ can be equally distant from AB and CD.

Ex. 2. *To find the locus of a point which moves so that its distances from two fixed points,* A *and* B, *are equal to one another.*



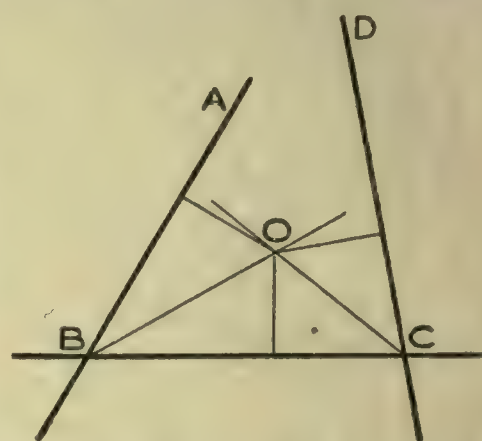
Evidently the locus sought is the line PN, which bisects at right angles the line joining A and B.

It is left to the student to

show that, whatever be the position of **P** on **NP**, **PA** = **PB**.

Also, to show that no point outside of **PN** can be equally distant from **A** and **B**.

Ex. 3. *To find a point which is equidistant from three given straight lines.*



Let **AB**, **BC**, **CD** be the three given straight lines.

Then if **BO** bisect the angle between **AB** and **BC**, **BO** is (Ex. 1.) ~~the~~ a locus of points equidistant from **AB** and **BC**.

Also, if **CO** bisect the angle between **BC** and **CD**, **CO** is ~~the~~ a locus of points equidistant from **BC** and **CD**.

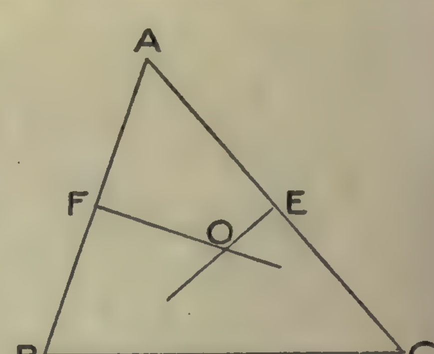
Hence **O**, the intersection of **BO** and **CO**, is a point equidistant from **AB**, **BC** and **CD**.

What other locus than BO is there of points equidistant from AB and BC?

What other locus than CO is there of points equidistant from BC and CD?

Discover three points in addition to O, which are equidistant from AB, BC, CD.

Ex. 4. *To find a point which is equidistant from three given points.*



Let **A**, **B**, **C** be the three given points.

Then if **FO** bisect **AB** at right angles, **FO** is (Ex. 2.) the locus of all points equidistant from **A** and **B**.

Also, if **EO** bisect **AC** at right angles, **EO** is the locus of all points equidistant from **A** and **C**.

Hence O, the intersection of FO and EO, is a point equidistant from A, B and C.

If a circle be described with centre O and radius OA, OB, or OC, it will pass through the three points A, B and C, **and be described about the triangle ABC.**

Exercises.

1. In Exercise 3 above, show that, if a line be drawn from O to the intersection of AB and CD, it bisects the angle between AB and CD.

It follows that the lines bisecting the angles of a triangle meet in a point.

2. In Exercise 4 above, show that, if a perpendicular be dropped from O on BC, it bisects BC.

It follows that the lines bisecting at right angles the sides of a triangle meet in a point.

3. Right-angled triangles are described on a given straight line as hypotenuse. Find the locus of the intersection of the lines which form the right angle.

4. OX, OY are two fixed straight lines at right angles. Points A and B are taken on OX, OY respectively, such that AB is of constant length. Find the locus of the middle point of AB.

5. Find the locus of the centre of a circle of given radius which rolls on the outside of a given circle. (The line joining the centres of two circles in contact passes through the point of contact.)

6. Find the locus of the centre of a circle of given radius which rolls on the inside of a given circle.

7. A point is subject to the conditions, (1) that it lies on a given line XY; (2) that it is equidistant from two fixed points A and B. Find the position of the point.

8. A point is subject to the conditions, (1) that it lies on the circumference of a given circle; (2) that it is equidistant from two fixed points A and B within the circle. Find the positions of the point. If A, B lie without the circle, can the point always be found?

9. A point is subject to the conditions, (1) that it lies on a given line XY; (2) that it is equidistant from the given straight lines AB and CD. Find the point's position.

Angles of a Polygon.

In § 31 of the Introduction it was shown that,—

The three angles of any triangle are together equal to two right angles.

Two angles of a triangle are together less than two right angles.

The exterior angle of any triangle is equal to the sum of the two interior and opposite angles, and therefore is greater than either of them.

The following, some of which have already been used, are obvious consequences of the first of these truths,—

If two triangles have two angles of the one respectively equal to two angles of the other, then the third angle of the one is equal to the third angle of the other.

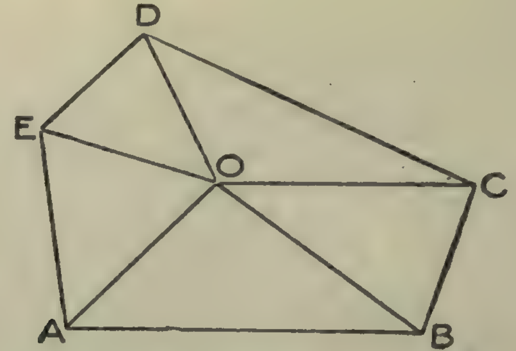
At least two angles of every triangle are acute.

In any right-angled triangle the two acute angles are complementary.

If one angle of a triangle is equal to the sum of the other two, the triangle is right-angled.

The sum of the angles of any quadrilateral figure is equal to four right angles, for it can be divided into two triangles.

The following is an important corollary to § 31, Introduction:



The sum of all the interior angles of a polygon of n sides is equal to $2n-4$ right angles.

For any polygon **ABCD..** of n sides may be divided up into n triangles.

And the angles of each triangle are equal to 2 right angles.

Therefore the angles of the n triangles are equal to $2n$ right angles.

Of these angles, the angles at **O** make up 4 right angles.

Therefore the sum of the interior angles of the polygon is $2n - 4$ right angles.

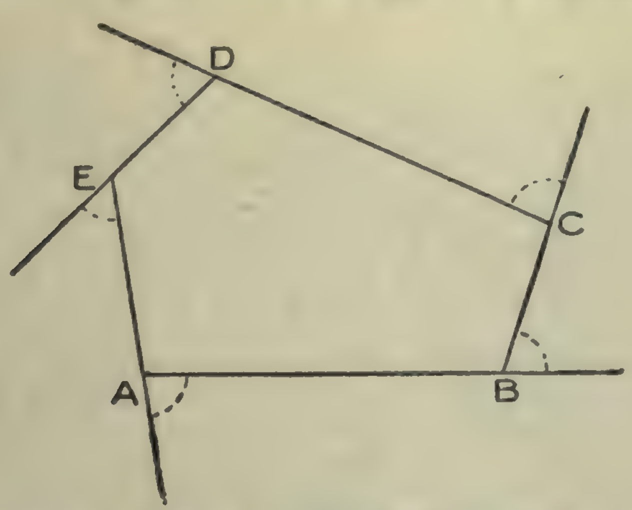
A **regular** polygon is one which has all its sides equal, and also all its angles equal.

If a regular polygon have n angles, the magnitude of each angle is evidently

$$\frac{2n-4}{n} \text{ right angles.}$$

which has no re-entrant an

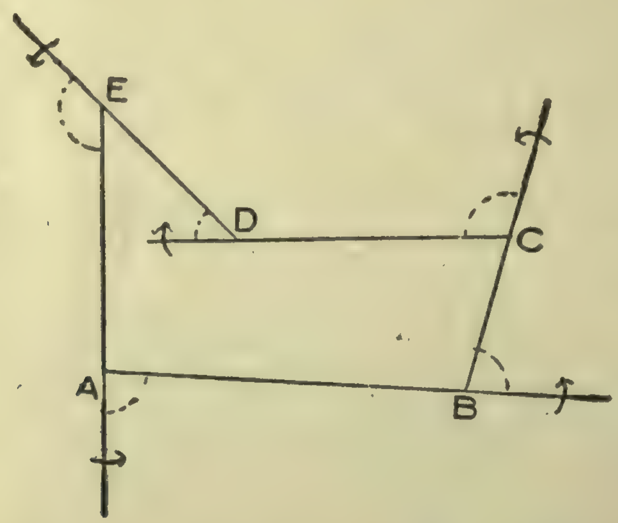
If the sides of any polygon be produced in order, then all the exterior angles so formed are together equal to 4 right angles.



For suppose a line to start from the position **AB**, and to rotate in succession through the exterior angles marked at **B, C, . . ,** into the positions **BC, CD, ...** On returning to the position **AB**, it has evidently made a complete revolution; *i.e.*, has turned through 4 right angles. Hence the sum of the exterior angles is 4 right angles.

Since the sum of both exterior and interior angles at A, B, C, . . is $2n$ right angles; therefore the sum of the interior angles is $2n-4$ right angles, as before proved.

The preceding proposition continues to hold, even where re-entrant angles occur in the polygon. In this case, however, we must consider the exterior angle at



the re-entrant angle (D) to be negative, for there the rotation referred to in the demonstration of the proposition, is evidently in a direction contrary to the other rotations. If therefore we wish to word the proposition so as to admit of no exception, we may say that the *algebraic* sum of the exterior angles is equal to 4 right angles.

Evidently the exterior angle of any regular polygon is $\frac{4}{n}$ of a right angle.

Exercises.

1. From the fact that the sum of the exterior angles of a polygon is 4 right angles, deduce the sum of the interior angles of the polygon.

2. Find the number of degrees in the angle of a regular pentagon; also the number of degrees in the exterior angle of a regular pentagon.

3. Find the number of degrees in the angle of a regular hexagon; also the number of degrees in the exterior angle of a regular hexagon.

4. If alternate sides of a polygon be produced to meet, show that the sum of the angles at the vertices so formed is $2n-8$ right angles.

5. If the bisectors of the angles B and C, of the triangle ABC, meet at O, show that the angle $BOC = 90^\circ + \frac{1}{2}A$.

6. If the bisectors of the exterior angles at B and C, of the triangle ABC, meet at O, show that the angle $BOC = 90^\circ - \frac{1}{2}A$.

7. The acute angle between any two straight lines is equal to the acute angle between any two straight lines at right angles to the former.

8. In a triangle ABC, if perpendiculars, dropped from B and C on the opposite sides, meet at O, show that the angle $BOC = B + C$.

9. If ABCD be any quadrilateral, and the bisectors of the angles at B and C meet at O, show that the angle $BOC = \frac{1}{2}(A + D)$.

10. How many sides has a regular polygon, if each exterior angle be 40°?

11. The angles at the base of an isosceles triangle are each double the vertical angle. How many degrees are there in each of the angles?

12. Construct a quadrilateral ABCD, such that $A + C = 180^\circ$, and therefore, of course $B + D = 180^\circ$. Show that the exterior angle at any corner is equal to the interior angle at the opposite corner.

13. In a triangle ABC, AD is perpendicular to BC, and AE bisects the angle at A. Show that the angle DAE is equal to half the difference between the angles at B and C.

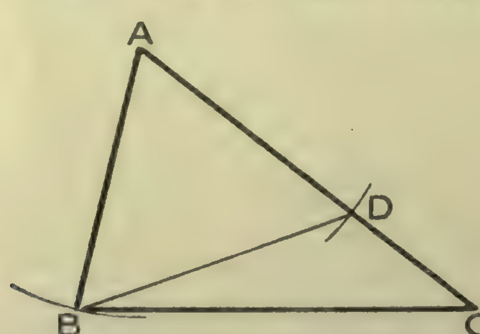
14. Through a point A, outside a given straight line BC, draw a straight line which shall make with BC an angle equal to a given angle. (Drawing a line parallel to another not permitted.)

Certain Geometrical Inequalities.

The next five theorems are illustrations of geometrical inequalities. We have already had examples of such. Thus, from In., § 6, we see that any two sides of a triangle are together greater than the third side, since a straight line is the shortest distance between its extreme points. Again, in § 31 of In., it was shown that the exterior angle of any triangle is greater than either of the interior and opposite angles; and also that any two angles of a triangle are together less than two right angles.

PROPOSITION XIII. THEOREM.

If one side of a triangle be greater than another side, the angle opposite the greater side is greater than the angle opposite the less.



Let ABC be a △ having AC > AB.

Then the ∠ABC *is* > ∠ACB.

From AC cut off AD = AB.

Join BD.

Then exterior ∠ADB is > the interior and opposite ∠ACB. (In., § 31.)

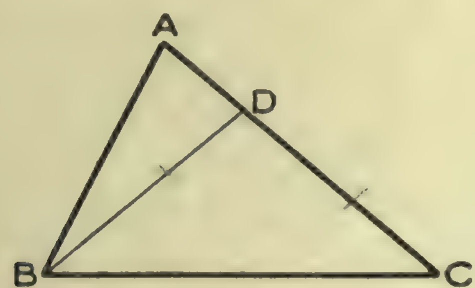
But ∠ABD = ∠ADB, because AB = AD; (Prop. 2.)

∴ ∠ABD > ∠ACB;

still greater therefore is ∠ABC than ∠ACB.

PROPOSITION XIV. THEOREM.

If one angle of a triangle be greater than another, the side opposite the greater angle is greater than the side opposite the less.



Let ABC be the $\triangle$ having $\angle ABC > \angle ACB$.

Then the side AC *is* $>$ *the side* AB.

At B make $\angle CBD = \angle ACB$. (Prop. 5.)

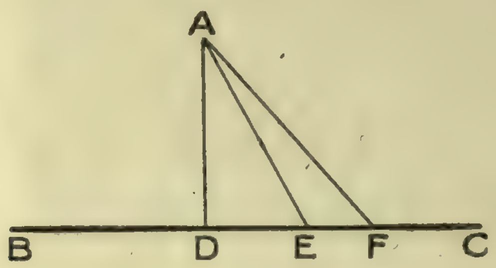
Then $DB = DC$; (Prop. 3.)

$\therefore$ $AC = AD + DC = AD + DB$, which is $> AB$. (In., § 6.)

That is, $AC > AB$.

NOTE. This Proposition is the converse of Prop. XIII.

COROLLARY: **The perpendicular is the shortest line that can be drawn from a given point to a given line; and a line nearer to the perpendicular is less than one more remote.**



For if $\angle$**ADE** be a rt. $\angle$, then $\angle$**AED** is $<$ rt. $\angle$.

Hence **AD**$<$**AE.**

Also $\angle$**AEF** is an obtuse $\angle$, since it is $>$ $\angle$**ADE**; (In., § 31.)

$\therefore$ $\angle$**AEF**$>$ $\angle$**AFE,** and **AF**$>$**AE.**

Evidently only two equal straight lines can be drawn from **A** to **BC**; *i.e.*, on opposite sides of **AD,** and making equal angles with it.

This corollary shows why we speak of the perpendicular from a point to a line, as *the distance* from the point to the line.

Exercises.

1. In a right-angled triangle the hypotenuse is the greatest side.

2. If one side of a triangle be less than another side, the angle opposite the smaller side must be an acute angle.

3. ABCD is a quadrilateral of which AD is the longest side, and BC the shortest. Show that the angle ABC is greater than the angle ADC; and the angle BCD greater than the angle BAD.

4. ABC is a triangle, and the angle A is bisected by AD, meeting BC in D. Show that BA is greater than BD, and CA greater than CD.

5. Every straight line drawn from the vertex of a triangle to the base, is less than the greater of the two sides, or than either of them if they be equal.

6. The greatest side of any triangle makes acute angles with each of the other sides.

7. The angles ABC, ACB of the triangle ABC are bisected by OB, OC. If AB be greater than AC, then OB is greater than OC.

8. In triangle ABC, side AB is greater than side AC. The angle A is bisected by a line meeting BC at D. Show that BD is greater than CD. (From AB cut off AE equal to AC, and join ED.)

The following are consequences of the fact that any two sides of a triangle are together greater than the third side.

9. Show that the sum of the diagonals of any quadrilateral figure is less than the sum of the sides.

10. Show that the sum of the diagonals of any quadrilateral figure is greater than half the sum of the sides.

11. The sum of the diagonals of a quadrilateral is less than the sum of any four lines that can be drawn from any point (except the intersection of the diagonals) to the four angular points.

12. If O be any point within the equilateral triangle ABC, show that any two of the straight lines OA, OB, OC are together greater than the third.

13. The difference of any two sides of a triangle is less than the third side.

14. In any triangle any two sides are together greater than twice the median which bisects the remaining side. (Produce median a distance equal to it; join end to either extremity of base of triangle.)

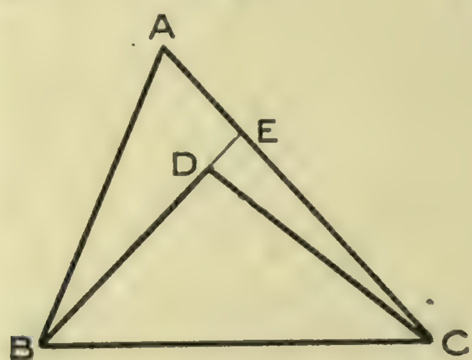
5. In any rectilineal figure the sum of the distances of any point from the angular points of the figure is greater than half the perimeter.

16. The sum of the two diagonals of any quadrilateral is greater than that of either pair of opposite sides.

17. In the triangle ABC, the line AD bisects the angle at A and meets BC at D. Show that the difference between AB and AC is greater than the difference between BD and DC.

PROPOSITION XV. THEOREM.

If from the ends of one side of a triangle there be drawn two straight lines to a point within the triangle, these two straight lines are together less than the sum of the two other sides of the triangle.



Let **ABC** be a $\triangle$, and from **B, C,** the ends of the side **BC,** let **BD, DC** be drawn to a pt. **D** within the $\triangle$.

Then the sum of **BD, DC** *is less than the sum of* **BA, AC.**

Produce **BD** to meet **AC** in **E.**

In $\triangle$**BAE,** $BA + AE > BE$.

To each add **EC.**

Then $BA + AC > BE + EC$.

Again, in $\triangle$**CED,** $CE + ED > CD$.

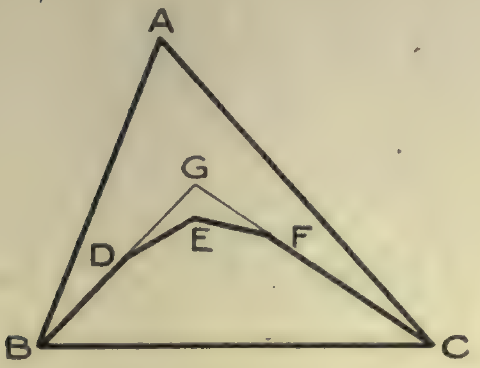
To each add **DB.**

Then $BE + EC > BD + DC$.

But it has been shown that

$BA + AC > BE + EC$;

$\therefore$, *a fortiori,* $BA + AC > BD + DC$.



COROLLARY: If lines **BD, DE, EF, FC** be drawn as in the figure, *there being no re-entrant angle*, it at once appears that **BA + AC > BG + GC > BD + DE + EF + FC.** Whatever be the number of lines **BD, DE, ... ,** the proof applies.

Exercises.

1. In the Proposition, show that the angle BDC is greater than the angle BAC.

2. In the Corollary, show that the angle between any two of the lines within the triangle is greater than the angle BAC.

3. The sum of the distances of any point within a triangle from its angular points is less than the perimeter of the triangle.

4. In the Proposition, show that the difference between BA + AC and BD + DC is less than twice AD.

5. Prove that the angle BDC is greater than the angle BAC, by joining AD and producing it.

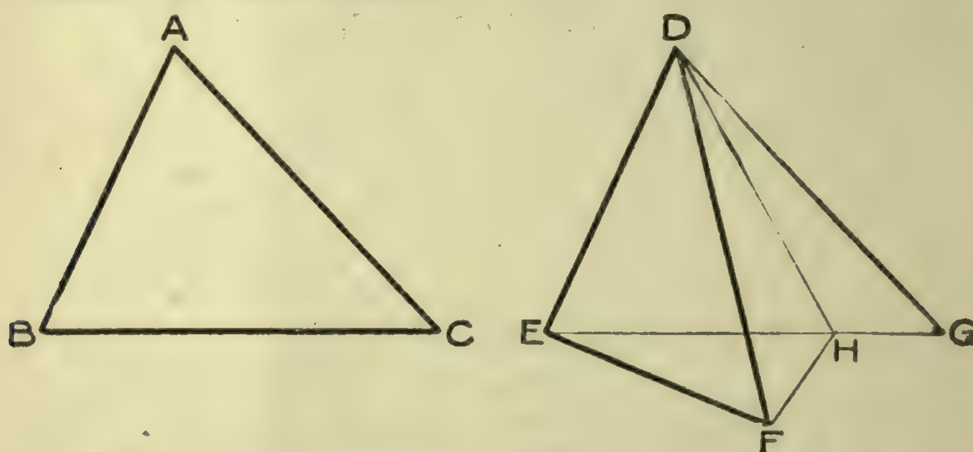
6. If two triangles be on the same base and on the same side of it, and have equal vertical angles, the vertex of each triangle must be outside the other triangle.

7. Show that the perimeter of a triangle is greater than that of any triangle which can be formed by joining any three points within it.

8. Two rectilineal figures are upon the same base, one of them being entirely within the other and having no re-entrant angles. Show that the outer figure has the greater perimeter.

PROPOSITION XVI. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of the one greater than the angle contained by the two sides of the other; then the base of that which has the greater angle is greater than the base of the other.



Let ABC, DEF be two $\triangle$s, having $AB = DE$, $AC = DF$; but the $\angle BAC >$ the $\angle EDF$.

Then the base BC *is* $>$ *the base* EF.

Apply the $\triangle ABC$ so that A falls on D, and AB on DE.

Then B coincides with E, because $AB = DE$.

Let C fall at G.

If F falls on EG, then evidently $EG > EF$ and $\therefore$ $BC > EF$.

But if not, let DH bisect $\angle FDG$, and meet EG in H.

Join FH.

Then in $\triangle$s FDH, GDH,

$DF = DG$,

DH is common to $\triangle$s,

$\angle FDH = \angle GDH$;

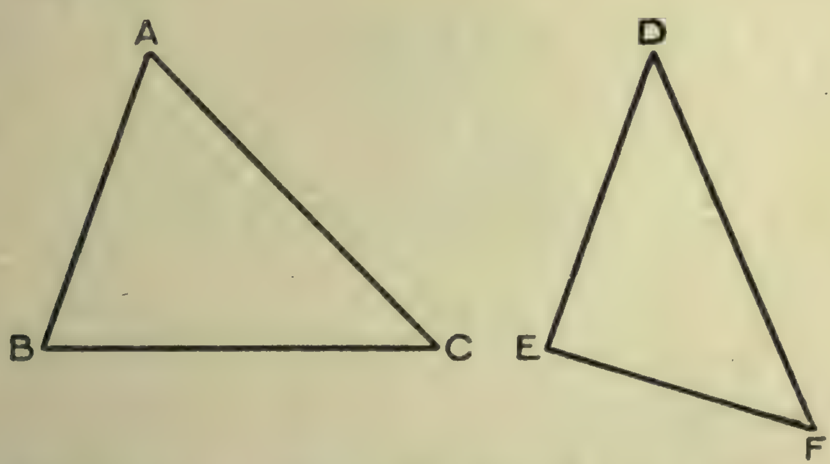
$\therefore$ $FH = GH$. (Prop. 6.)

Then $EH + HG = EH + HF$, which is $> EF$.

That is, EG (or BC) is $> EF$.

PROPOSITION XVII. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the base of the one greater than the base of the other; then the angle contained by the sides of that which has the greater base is greater than the angle contained by the sides of the other.



Let **ABC**, **DEF** be two $\triangle$s, having **AB** = **DE**, **AC** = **DF**, but **BC** > **EF**.

Then the $\angle$**BAC** > *the* $\angle$**EDF**.

For if $\angle$**BAC** be not > $\angle$**EDF**,

it must be either equal to it or less.

But $\angle$**BAC** cannot be = $\angle$**EDF**,

for then **BC** would be = **EF**, (Prop. **6**.)

which is contrary to hypothesis.

Nor can $\angle$**BAC** be < the $\angle$**EDF**,

for then **BC** would be < **EF**, (Prop. **16**.)

which also is contrary to hypothesis.

$\therefore$ the $\angle$**BAC** > the $\angle$**EDF**.

Exercises.

1. ABCD is a quadrilateral having AB and CD equal, but the diagonal BD greater than the diagonal AC. Prove that the angle BCD is greater than the angle ABC.

2. ABC is a triangle having the angle ABC greater than the angle ACB. If AD be drawn to the middle point of BC, show that the angle ADC is obtuse.

3. ABC is a triangle having AB less than AC, and P is any point in the line joining A to the middle point of BC. Show that P is nearer to B than to C.

4. ABC is a triangle having the sides AB, AC fixed in length, and AB being greater than AC. The side AB remaining fixed in position, draw the locus of C as the angle BAC varies from 0° to 180°.

What statement can you make as to the changes in the length of a line drawn from a fixed point without a circle to a point on the circumference, as the position of the latter point varies?

5. The same as Exercise 4, but with AB less than AC.

In the second part, the fixed point is now within the circle.

6. If one chord of a circle be greater than another, the angle subtended at the centre by the former is greater than the angle subtended at the centre by the latter.

7. State and prove the converse of Exercise 6.

8. The side AB of the triangle ABC is greater than the side AC. From BA, CA equal parts BD, CE are cut off. Show that BE is greater than CD.

BOOK II.

PARALLELISM.

PARALLEL LINES.

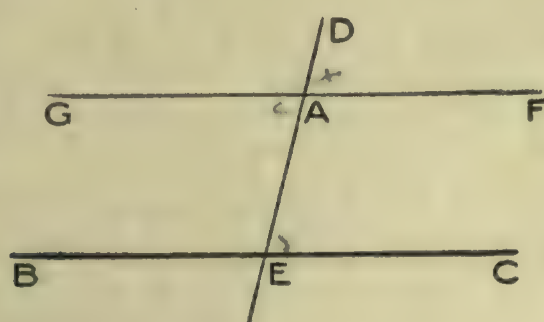
EQUIVALENCE OF AREAS.

Parallel Lines.

PROPOSITION I. PROBLEM.

q.i.

To draw a straight line through a given point, parallel to a given straight line.



Let **A** be the given point, and **BC** the given st. line.

It is required to draw through **A** *a st. line* ∥ *to* **BC**.

Through **A** draw any st. line, **DAE**, cutting **BC** in **E**.

At the pt. **A** in the st. line **DAE**, make the ∠**DAF** = the ∠**AEC**, and exterior to it.

Produce **FA** to **G**.

Then the st. line **GAF**, drawn through **A**, is ∥ to **BC**.

For the st. line **DAE**, falling on the st. lines **GF**, **BC**, makes the ext. ∠**DAF** = the int. and opp. ∠**AEC**;

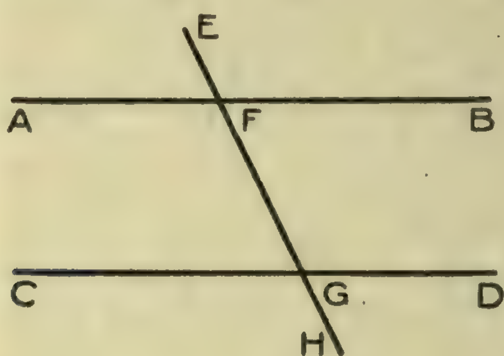
∴ **GAF**, drawn through **A**, is ∥ to **BC**. (In., § 29.)

Angles situated as **GAE**, **AEC**, are called **alternate** angles.

Angles situated as **FAE**, **AEC**, are spoken of as the **two interior angles on the same side** (of the transversal **DAE**.)

PROPOSITION II. THEOREM.

If a straight line fall on two parallel straight lines, it makes (1) the alternate angles equal, and (2) the two interior angles on the same side together equal to two right angles.



Let the st. line **EH** fall on the || st. lines **AB, CD.**

Then (1) the alternate ∠s **AFG, FGD** *are equal; and* (2) *the two interior* ∠s **BFG, FGD** *on the same side are together* = *two rt.* ∠s.

(1) Because **AB** is || to **CD**, and **EH** falls upon them, the ext. ∠**EFB** is = int. and opp. ∠**FGD**. (In., § 29.)

But ∠**AFG** is = vertically opp. ∠**EFB**; (In., § 13.)

∴ ∠**AFG** = ∠**FGD**.

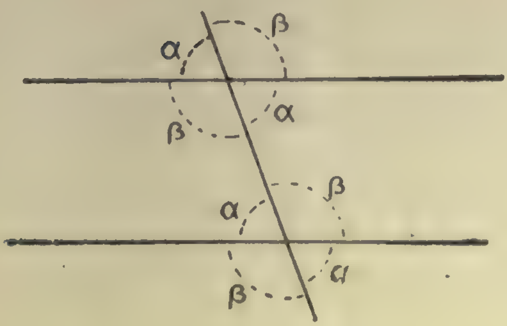
(2) Because **AB** is || to **CD**, and **EH** falls upon them, the ext. ∠**EFB** is = int. and opp. ∠**FGD**.

To each add the ∠**BFG**.

Then ∠**EFB** + ∠**BFG** = ∠**BFG** + ∠**FGD**.

But ∠**EFB** + ∠**BFG** = two rt. ∠s; (In., § 18.)

∴ ∠**BFG** + ∠**FGD** = two rt. ∠s.



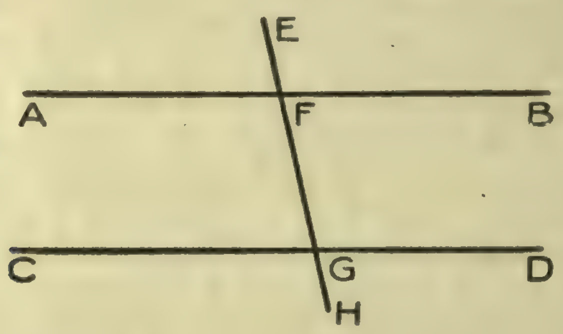
In the preceding diagram all the angles marked α are equal to one another; and also all the angles marked β are equal to one another.

Moreover we have, in all cases,

$$\alpha + \beta = 180°.$$

PROPOSITION III. THEOREM.

If a straight line, falling on two other straight lines, make (1) the alternate angles equal to one another; or (2) the two interior angles on the same side together equal to two right angles, then the two straight lines are parallel.



Let the st. line **EH**, falling on the two st. lines **AB**, **CD**, make (1) the alt. ∠s **AFG**, **FGD** equal, or (2) the two int. ∠s **BFG**, **FGD**, on the same side, together = 2 rt. ∠s.

Then **AB** *is* ∥ *to* **CD**.

(1) Because ∠**AFG** = ∠**FGD**, (Hyp.)

and ∠**AFG** = ∠**EFB**; (In., § 13.)

∴ ∠**EFB** = ∠**FGD**.

Hence **AB** is ∥ to **CD**. (In., § 29.)

(2) Again, because ∠**BFG** + ∠**FGD** = 2 rt. ∠s, (Hyp.)

and ∠**EFB** + ∠**BFG** = 2 rt. ∠s; (In., § 18.)

∴ ∠**EFB** + ∠**BFG** = ∠**BFG** + ∠**FGD**.

Take away the common ∠**BFG**,

and ∠**EFB** = ∠**FGD**.

Hence **AB** is ∥ to **CD**. (In., § 29.)

Exercises.

1. Straight lines which are perpendicular to the same straight line are parallel to one another.

2. If a straight line meet two or more parallel straight lines, and be perpendicular to one of them, it is also perpendicular to the others. Show also that the perpendicular distance between the same two parallel lines is constant.

3. If the arms of two angles be parallel, each to each, the angles are either equal or supplementary.

4. Two straight lines AB, CD bisect one another at O. Show that AC, BD are parallel, and also AD, CB.

5. A line is intercepted by two parallel lines and is bisected at P. Show that any other line through P, and intercepted by the parallel lines, is bisected at P.

6. If a quadrilateral have its angles right angles, show that its opposite sides are parallel.

7. **Show that the opposite sides of a rhombus are parallel.**

8. Construct a parallelogram, having given one of its angles, and the lengths of the sides which are the arms of this angle.

9. **If one angle of a parallelogram be a right angle,** *i.e.*, **if it be a rectangle** (see Definition), **show that all its angles are right angles.**

10. **Construct a square according to its definition, and show that its opposite sides are parallel, and that all its angles are right angles.**

11. AD bisects the angle BAC; and E is a point in AB such that AE = ED. Show that ED is parallel to AC.

12. Show that if two straight lines be drawn from two of the angular points of a triangle to the opposite sides, these two straight lines cannot bisect each other.

13. If the straight line which bisects the exterior angle of a triangle be parallel to the opposite side, the triangle is isosceles.

14. Any straight line drawn parallel to the base of an isosceles triangle makes equal angles with the sides.

For additional Exercises, see p. 88.

Notes.

1. "Since parallel lines have the same direction, they each deviate by the same amount from any other direction" (In., § 29). In making this statement we assume that directions which are not the same, intersect; *i.e.*, that **any two lines which are not parallel intersect,** such lines being produced, if necessary.

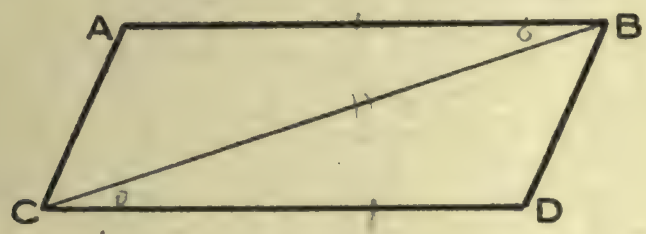
2. If a transversal fall across two lines which are not parallel, since the four interior angles at the two points of intersection are together equal to four right angles, the interior angles on one side of the transversal are together less, and those on the other side are together greater than two right angles; for if on either side they were equal to two right angles the lines would be parallel. (Prop. 3, Bk. II.)

Such lines evidently intersect on that side of the transversal on which are the angles together less than two right angles; otherwise we should have two angles of a triangle together greater than two right angles. (In., § 31.)

As exercises, the student may now prove that **BO, CO** (Ex. 3, p. 58) meet; also **EO, FO** (Ex. 4, p. 58); and later, when Prop. 13, Bk. II., is reached, he can show that **NA, LK** meet.

PROPOSITION IV. THEOREM.

Straight lines which join the extremities of equal and parallel straight lines towards the same parts, are themselves equal and parallel.



Let the equal and || st. lines **AB, CD** be joined toward the same parts by **AC** and **BD**.

Then **AC** *and* **BD** *are both equal and* ||.

Join **BC.**

Since **AB** is || to **CD**, the alt. ∠s **ABC, BCD** are equal.

Then in Δs **ABC, DCB,**

AB = CD,

BC is common to Δs,

∠ **ABC** = ∠ **BCD** ; (Prop. 2, Bk. II.)

∴ Δs are equal in all respects;

and **AC = BD.**

Also ∠ **ACB** = ∠ **DBC.**

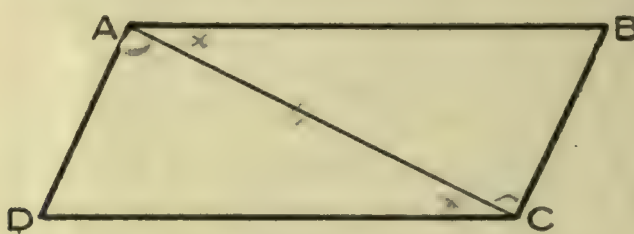
But these are alternate ∠s;

∴ **AC** is || to **BD.** (Prop. 3, Bk. II.)

PROPOSITION V. THEOREM.

Jil

The opposite sides and angles of a parallelogram are equal to one another, and each diagonal divides the parallelogram into two triangles equal in all respects.



Let **ABCD** be a $\|^m$, and **AC** its diagonal.

Then **AB** = **CD**, **AD** = **CB**, ∠**ABC** = ∠**CDA**, ∠**BAD** = ∠**DCB**; *and the* Δs **ABC, CDA** *are equal in all respects.*

In Δs **ABC, CDA**,

∠**BAC** = ∠**DCA**, being alt. ∠s, (Prop. 2, Bk. II.)

∠**BCA** = ∠**DAC**, " " ",

AC is common to Δs;

∴ Δs are equal in all respects. (Prop. 7, Bk. I.)

Hence **AB** = **CD**,

CB = **AD**,

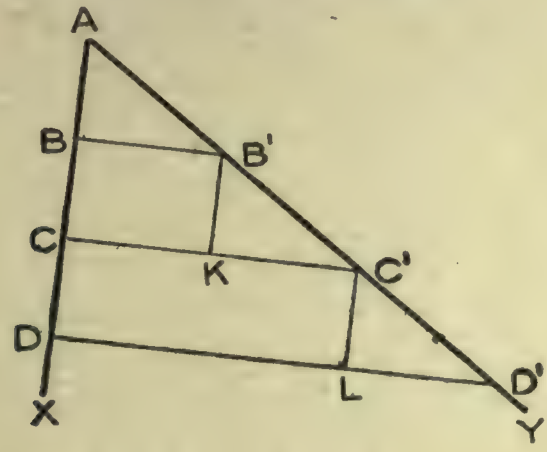
∠**ABC** = ∠**CDA**.

Also, since ∠**BAC** = ∠**DCA**,

and ∠**DAC** = ∠**BCA**;

∴ whole ∠**BAD** = whole ∠**DCB**.

COROLLARY 1: **If one of two lines be divided into equal segments, and through the points of section parallel lines be drawn intersecting the other line, the parallel lines divide this other line into equal segments.**



Let **AB, BC, CD** ... be equal to one another, and **BB′, CC′, DD′**, ... parallel lines.

Let **B′K, C′L**, .. be drawn parallel to **AD.**

Then **BK, CL**, ... are parallelograms, and the lines **AB, B′K, C′L**, ... are all equal.

Evidently the Δs, **ABB′, B′KC′, C′LD′**, ... are therefore equal in all respects.

Hence the lines **AB′, B′C′, C′D′**, ... are all equal to one another, though in general, of course, not equal to the segments **AB, BC**, ...

COROLLARY 2: The preceding suggests a means of dividing a given line into any number of equal parts.

For let **AD′** be the given line. Through **A** draw any line **AX**, and on it take equal lengths **AB, BC, CD.**

Join **DD′**, and through **B, C**, draw lines parallel to **DD′.**

Then by the preceding corollary the segments of **AD′** are equal.

Exercises.

1. Show that the diagonals of a parallelogram bisect each other.

2. Show that the diagonals of any rhombus (including a square) bisect each other at right angles.

3. Show that a quadrilateral is a parallelogram—

   (1) If one pair of opposite sides are equal and parallel.

   (2) If pairs of opposite sides are equal.

   (3) If pairs of opposite angles are equal.

   (4) If the diagonals bisect each other.

4. Show that the diagonals of a rectangle are equal.

5. Show that if the diagonals of a parallelogram are equal, it must be a rectangle; and that if, in addition, they are at right angles, it must be a square.

6. Show that any straight line through the intersection of the diagonals of a parallelogram divides the parallelogram into two equal parts.

7. Through a given point D within a given angle BAC, draw a straight line BDC, such that BD is equal to DC.

8. If a diagonal of a parallelogram bisects the angles through which it passes, the figure is a rhombus.

9. If two railway tracks of the same gauge intersect at any angle, the figure thus formed is a rhombus.

10. The diagonals of a rhombus bisect one another at right angles.

11. Draw a straight line DE parallel to the base BC of a triangle ABC, and terminated by the sides or sides produced, so that DE may be of given length.

12. If two opposite sides of a quadrilateral are parallel, and the other two sides equal but not parallel, prove that the angles adjacent to each of the parallel sides are equal.

13. Describe a rhombus having its angular points on the sides of a given parallelogram, such that one diagonal of the rhombus passes through a given point.

14. Show that if the bisectors of the angles of a quadrilateral form a rectangle, the quadrilateral must be a parallelogram.

15. AB, CD, EF are three parallel straight lines, ACE being a straight line, and also BDF. Show that if AC, CE are equal, so also are BD, DF.

16. Find axes of symmetry, and also a centre of symmetry for,—

(1) square; (2) rectangle; (3) rhombus.

What sort of symmetry do you discover in an ordinary parallelogram?

17. What symmetry do you discover in the quadrilateral ABCD, if $AB = AD$, and $CB = CD$?

18. ABC is a triangle, and through D, the bisection of AB, a line is drawn parallel to BC. Show that this line bisects AC also.

Conversely, show that the line joining the middle points of AB and AC is parallel to BC.

19. ABCD is a parallelogram, and E, F are the bisections of the sides BC, AD. Show that DE, BF trisect the diagonal AC.

20. If the middle points of adjacent sides of a quadrilateral be joined, the figure thus formed is a parallelogram.

21. The straight lines joining the middle points of the opposite sides of any quadrilateral bisect each other.

22. In Exercise 15, show that CD is an arithmetic mean between AB and EF, *i.e.*, that it is half their sum.

23. Any straight line drawn from the vertex of a triangle to the base, is bisected by the line which joins the middle points of the sides.

24. The three straight lines which join the middle points of the sides of a triangle, divide the triangle into four triangles which are equal in all respects, and each equiangular to the original triangle.

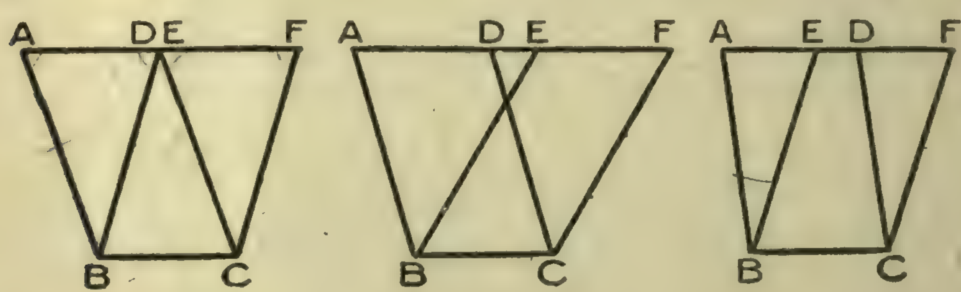
25. If through the angular points of a triangle, straight lines be drawn parallel to the opposite sides, the triangle so formed has its sides double the sides of the original triangle, and its angles equal to those of the original triangle.

Hence show (Ex. 2, p. 59) **that lines through the angular points of a triangle perpendicular to the opposite sides, meet in a point.**

Equivalence of Areas.

PROPOSITION VI. THEOREM.

Parallelograms on the same base and between the same parallels are equal in area.



Let **ABCD, EBCF** be two $\|^{ms}$ on the same base **BC**, and between the same $\|^{s}$ **AF, BC.**

Then these $\|^{ms}$ *are equal in area.*

In Δs **EAB, FDC,**

AB = **DC,** being opp. sides of a $\|^{m}$, (II., 5.)

int. $\angle$ **EAB** = ext. $\angle$ **FDC,** (In., § 29.)

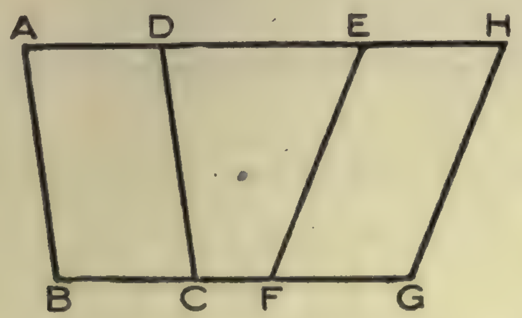
ext. $\angle$ **AEB** = int. $\angle$ **DFC;**

$\therefore$ Δ **EAB** = Δ **FDC;** (I., 7.)

$\therefore$ fig. **ABCF** less Δ **FDC** = fig. **ABCF** less Δ **EAB;**

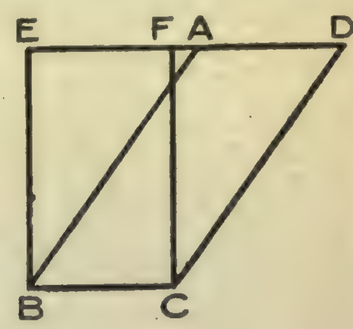
that is $\|^{m}$ **ABCD** = $\|^{m}$ **EBCF.**

COROLLARY: **Parallelograms on equal bases and between the same parallels are equal in area.**



For **BC** and **FG** being equal, if the $||^m$ **EFGH** be moved to the left, so that **FG** coincides with **BC**, the conditions of the Proposition itself are realized, and the parallelograms are equal.

AREA OF A PARALLELOGRAM. The $||^m$ **ABCD** and the rectangle **EBCF**, being on the same base and between the same parallels, are equal in area;



$\therefore$ area of **ABCD** = area of **EBCF**,

= **BC** × **BE**

= *base* × *altitude*.

Exercises.

1. Construct a rhombus equal to a given parallelogram, and having each of its sides equal to the longer side of the parallelogram.

2. In the preceding question, what condition is necessary that the rhombus may be constructed with each of its sides equal to the shorter side of the parallelogram?

3. Make a rectangle equal to a given parallelogram, and having one of its sides equal to a side of the parallelogram.

4. Show that if the lengths of the sides of a parallelogram be given, its area will be greatest when it is a rectangle.

5. Show that parallelograms with equal bases and equal altitudes are equal.

6. Show that equal parallelograms with equal bases must have equal altitudes.

7. Equal parallelograms with the same altitude must be on equal bases.

8. Divide a parallelogram into four equal parallelograms.

9. Construct a parallelogram equal to a given parallelogram, having one side equal to a side of the latter, and another side of given magnitude. What limit is there to the length of this latter side?

10. Construct a parallelogram equal to a given parallelogram, having one side equal to a side of the latter and an angle adjacent to this side of given magnitude.

11. ABCD is a parallelogram; AE, DF are drawn parallel to each other to meet BC, produced if necessary in E, F; AG, EH are drawn parallel to each other to meet DF, produced if necessary in G, H. Prove that the parallelograms ABCD, AEHG are equal in area.

To follow Exercises on p. 79.

✓ 12. If from any point in the straight line bisecting any angle, lines be drawn parallel to the arms of the angle, these lines are equal to one another, and the resulting figure is a rhombus.

✓ 13. Through each angular point of a triangle a straight line is drawn parallel to the opposite side. Show that the triangle formed by these lines is equiangular to the given triangle.

14. If two triangles have two angles of the one equal to two angles of the other, and be equal in area, then they are equal in all respects.

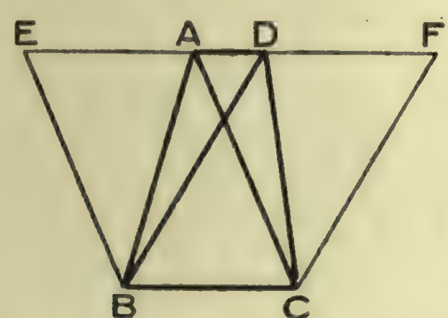
15. Find a point between two intersecting straight lines, such that perpendiculars from the point on the lines may be of given length.

16. Construct a triangle, having given the base, the altitude, and the length of the line from the vertex to the middle point of the base.

17. A straight line DE parallel to the base BC of a triangle ABC, makes the lengths BD, CE equal to one another. Show that the triangle ABC is isosceles.

PROPOSITION VII. THEOREM.

Triangles on the same base and between the same parallels are equal in area.



Let the Δs **ABC, DBC** be on the same base **BC**, and between the same $\|^s$ **AD, BC**.

Then these Δs are equal in area.

Produce **AD** both ways to **E** and **F**.

Through **B** draw **BE** $\|$ to **CA**; and through **C** draw **CF** $\|$ to **BD**. II. 1

Then the $\|^{ms}$ **EBCA, DBCF**, being on the same base **BC**, and between the same $\|^s$, are equal in area. (II., 6.)

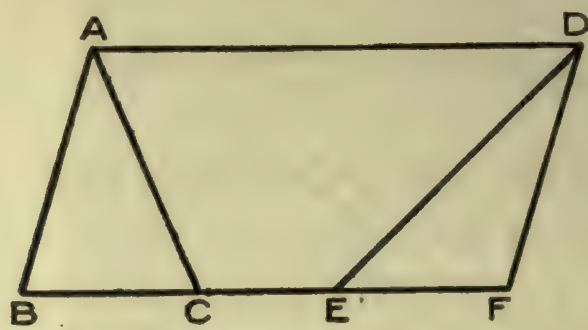
But Δ**ABC** is half $\|^m$ **EBCA**, (II., 5.)

and Δ**DBC** is half $\|^m$ **DBCF**;

$\therefore$ Δ**ABC** $=$ Δ**DBC**.

ML

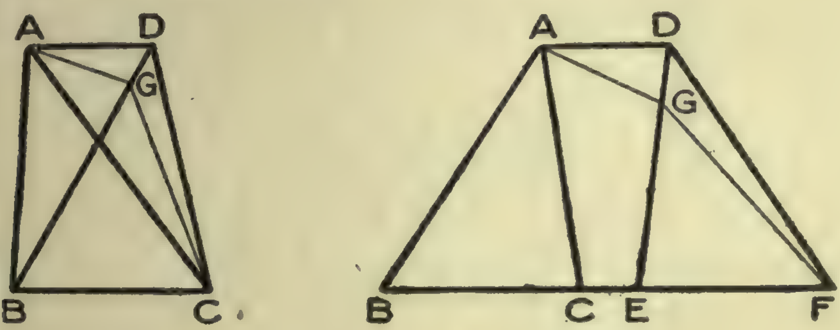
COROLLARY: **Triangles on equal bases and between the same parallels are equal in area.**



For **BC** and **EF** being equal, if the $\triangle$**DEF** be moved to the left, so that **EF** coincides with **BC**, the conditions of the Proposition itself are realized, and the triangles are equal in area.

PROPOSITION VIII. THEOREM.

Triangles, equal in area, on the same base, or on equal bases that are in the same straight line, and on the same side of the base line, are between the same parallels.



Let **ABC** and **DBC** (or **DEF**) be two $\triangle$s equal in area, on the same base **BC**, or on equal bases **BC, EF** that are in the same st. line, and on the same side of the base line.

Then these $\triangle$s *are between the same* $\parallel^s$*; that is,* **AD** *is* $\parallel$ *to* **BC.**

For if **AD** be not $\parallel$ to **BC**, through **A** draw **AG** $\parallel$ to **BC**, meeting **DB** (or **DE**) in **G**.

Then $\triangle$s **ABC, GBC** (or **GEF**) are equal in area. (II., 7.)

But the $\triangle$s **ABC, DBC** (or **DEF**) are equal in area; (Hyp.)

$\therefore$ $\triangle$s **GBC, DBC** (or **GEF, DEF**) are equal in area:

a part equal to the whole, which is impossible.

Hence **AG** cannot be $\parallel$ to **BC.**

Similarly it can be shown that no other st. line through **A**, except **AD**, is $\parallel$ to **BC**;

$\therefore$ **AD** is $\parallel$ to **BC.**

Exercises.

1. Find the locus of the vertex of a triangle whose base is given and area constant.

2. If two triangles have equal altitudes and equal areas, their bases must be equal.

A median is a line drawn from any angle of a triangle to the bisection of the opposite side.

3. Show that the median of a triangle divides it into two parts equal in area.

4. In the triangle ABC, BE and CF, the medians through B and C, intersect in O. Prove the following:

   (1) Triangles FBC, ECB are equal.

   (2) Triangles OFB, OEC are equal.

   (3) Triangles OFB, OFA are equal.

   (4) Triangles OFA, OEA are equal.

   (5) If D be the bisection of BC, then AO, OD are in the same straight line.

Hence the three medians of a triangle pass through the same point.

5. Prove that a parallelogram is divided by the diagonals into four equal areas.

6. ABC is a triangle, and AD the median through A. If E be any point in AD, show that the triangles EAB, EAC are equal in area.

7. Prove by means of Props. VII. and VIII. that the straight line joining the bisections of the sides of a triangle is parallel to the base.

8. On the base of a given triangle construct an isosceles triangle equal in area.

9. If the middle points of the sides of any quadrilateral are joined in order, show that the parallelogram so formed is half the quadrilateral, the quadrilateral having no re-entrant angle.

10. Two triangles of equal area stand on the same base but on opposite sides of it. Show that the line joining their vertices is bisected by the base or base produced.

11. Two triangles have two sides of the one equal to two sides of the other, each to each, but the angles contained by these sides supplementary. Prove that the triangles are equal in area.

12. ABCD is a parallelogram, and P is any point in AC. Show that the triangles ABP, ADP are equal in area, and also the triangles CBP, CDP.

13. P is a point within the triangle ABC, such that the triangles APB, APC are equal in area. Show that AP produced bisects BC.

14. AB, AC are two intersecting lines, and D, E, any two points in AB. Draw DF, EG terminated by AC and intersecting in O, such that the triangles OED, OFG may be equal in area.

15. ABC is any triangle, and D any point in AB. Through D draw a line which shall divide the triangle into two parts equal in area.

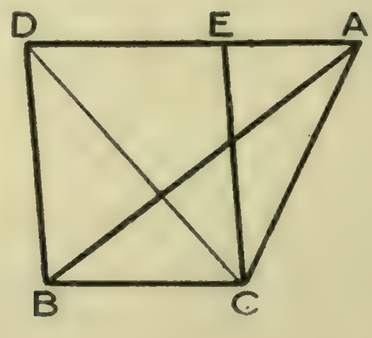
16. ABCD is a quadrilateral which is bisected by AC. Show that BD is bisected by AC.

17. If two triangles have the same altitude, but the base of one be a multiple of the base of the other, show that the former triangle is the same multiple of the latter.

18. If O be the point in which the medians AD, BE, CF of a triangle ABC intersect (Ex. 4), show that AO = 2OD, BO = 2OE, CO = 2OF.

PROPOSITION IX. THEOREM.

If a triangle and a parallelogram be on the same base and between the same parallels, the parallelogram is double of the triangle.



Let the $\triangle$ **ABC** and the $\|^m$ **DBCE** be on the same base **BC,** and between the same $\|^s$ **DA, BC.**

Then the $\|^m$ *is double the* $\triangle$.

Join **DC.**

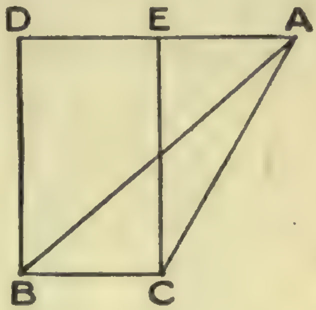
Then $\triangle$s **DBC, ABC** are equal in area. (II., 7.)

But $\|^m$ **DBCE** is double of $\triangle$**DBC**; (II., 5.)

$\therefore$ $\|^m$ **DBCE** is double of $\triangle$**ABC.**

COROLLARY. Evidently if the parallelogram and triangle be on equal bases and between the same parallels, the parallelogram is double of the triangle.

AREA OF A TRIANGLE. The triangle **ABC** and the rectangle **DBCE**, being on the same base, and between the same parallels, the rectangle is double of the triangle;



$$\therefore \text{area of } \mathbf{ABC} = \tfrac{1}{2} \text{ area of } \mathbf{DBCE}$$
$$= \tfrac{1}{2} \mathbf{BC} \times \mathbf{BD}$$
$$= \tfrac{1}{2} \text{ base} \times \text{altitude of } \triangle.$$

Exercises.

1. Show that in a right-angled triangle the rectangle contained by the two sides is equal to the rectangle contained by the hypotenuse and the perpendicular on it from the right angle.

2. O is any point within the parallelogram ABCD. Show that the sum of the triangles OAB, OCD is equal to the sum of the triangles OAD, OBC.

3. ABCD is a parallelogram, and P, Q any points in AB, BC, respectively. Show that the triangles PCD, QAD are equal in area.

4. ABC is a triangle and E, F are the middle points of AC, AB, respectively. Show that the triangle AFE is one quarter of ABC.

5. If two sides of a triangle be given in length, show that its area is greatest when they contain a right angle.

6. Show that the area of a rhombus is half the rectangle contained by its diagonals.

7. ABCD is a parallelogram whose diagonals intersect in E, and O is any other point in the parallelogram. Show that the difference between the triangles OAB, OBC is twice the triangle OBE.

8. ABCD is a quadrilateral having BC parallel to AD, and E is the middle point of DC. Show that the triangle AEB is half the quadrilateral.

9. On the same side of the straight line ABC equal rectangles ABDE, ACFG are described. Show that BG and FD are parallel.

10. In the preceding question, what additional condition must be introduced, that BG and FD may be parallel, if parallelograms be substituted for rectangles?

11. Through the angular points of a quadrilateral straight lines are drawn parallel to the diagonals. Show that the parallelogram so formed is double the quadrilateral in area.

12. O is any point within the parallelogram ABCD. Show that the sum of the areas of the triangles OAB, OCD is half the area of the parallelogram.

13. From any point within a given equilateral triangle, perpendiculars are let fall on the three sides. Show that the sum of these perpendiculars is constant for the same equilateral triangle.

14. If the sides AB, AD of a rhombus be bisected in E, F, show that the area of the triangle CEF is three-eighths of the area of the rhombus.

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15. ABCD is a parallelogram, and EF, parallel to BD, cuts CB, CD in E and F respectively. Show that the triangles ABE, AFD are equal in area.

16. If through the vertices A, B, C of a triangle parallel lines be drawn intersecting the sides opposite these points in D, E, F, show that the area of the triangle DEF is twice that of ABC.

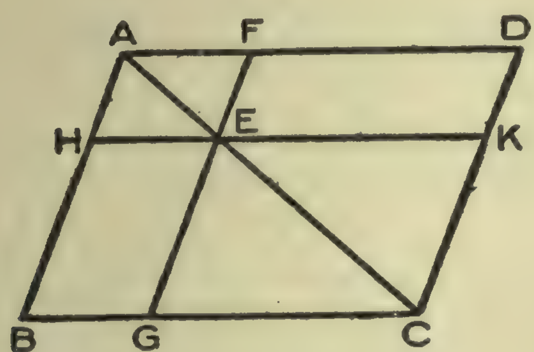
17. Any parallelograms ABDE, ACFG are described externally on the sides AB, AC of a triangle ABC, and DE, FG meet in H. On BC a parallelogram BKLC is constructed, having its sides BK, CL equal and parallel to AH. Show that BKLC is in area equal to the sum of the areas of ABDE and ACFG.

18. If P be a point within the triangle ABC, such that the sum of the areas of the triangles PAB, PAC is constant, prove that the locus of P is a straight line.

What does this proposition become when the point P moves along its locus outside the triangle ABC?

PROPOSITION X. THEOREM.

The complements of the parallelograms about the diagonal of any parallelogram are equal in area.



Let **ABCD** be a $\|^m$, and **BE, ED** the complements of the $\|^{ms}$ about the diagonal **AC**, formed by drawing **FEG, HEK** $\|$ to **AB, AD**, respectively, through any point **E** on **AC**.

Then the complements **BE, ED** *are equal in area.*

Since **AHEF** is a $\|^m$ and **AE** its diagonal;

$\therefore$ △**AHE** = △**AFE**. (II., 5.)

Since **EGCK** is a $\|^m$ and **EC** its diagonal;

$\therefore$ △**EGC** = △**EKC**.

Hence △**AHE** + △**EGC** = △**AFE** + △**EKC**.

But whole △**ABC** = whole △**ADC**;

$\therefore$ the remainder, the complement **BE**,

= the remainder, the complement **ED**.

Exercises.

1. In the figure of the Proposition, show that the triangles AED, AEB are equal in area.

Also that the triangles FHG, HFK are equal in area.

2. Show also that HF and GK are parallel.

3. Where must E be, that the parallelograms HF, GK may be equal?

4. Prove the converse of the Proposition,—that if the areas of BE, ED are equal, then E must lie on AC.

5. In the Proposition, show that if ABCD be a rhombus, so also is each of the figures HF, GK; also that if ABCD be a square, then are HF, GK squares.

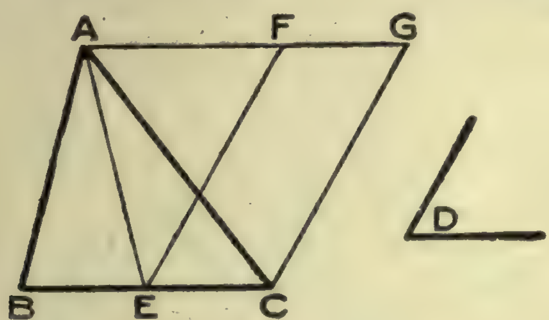
6. If ABCD in the Proposition, be a rhombus, show that BE, ED are not only equal in area, but can be made to coincide with one another, *i.e.*, are congruent.

7. In the figure of the Proposition, show that the triangles HEC, FEC are equal in area.

8. In the same figure, show that BD and GK are parallel. (Both are bisected by AC.)

PROPOSITION XI. PROBLEM.

To describe a parallelogram equal to a given triangle, and having an angle equal to a given angle.



Let **ABC** be the given $\triangle$, and **D** the given $\angle$.

It is required to construct a $\|^m$ *equal to* **ABC**, *and having an* $\angle$ *equal to* **D**.

Bisect **BC** at **E**.

Join **AE**.

At the point **E** in **EC** make the $\angle$**CEF** = the $\angle$**D**.

Through **A** draw **AFG** $\|$ to **BC**, and through **C** draw **CG** $\|$ to **EF**.

Then **ECGF** is the required $\|^m$.

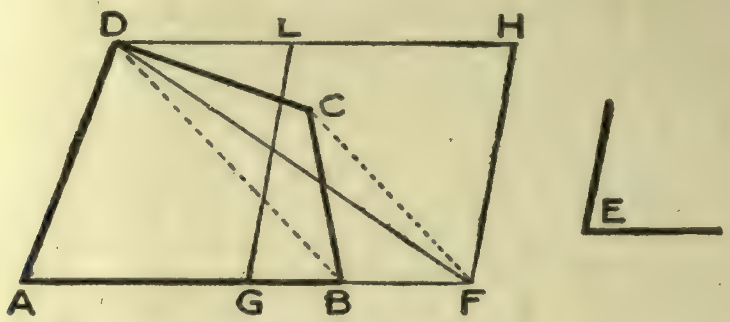
Because $\triangle$s **ABE, AEC** are on equal bases **BE, EC**, and between the same $\|^s$, they are equal (II., 7, Cor.); and $\triangle$**ABC** is double of $\triangle$**AEC**.

Also, because $\triangle$**AEC** and $\|^m$ **ECGF** are on the same base **EC** and between the same $\|^s$, the $\|^m$ **ECGF** is double of $\triangle$**AEC**. (II., 9.)

$\therefore$ $\|^m$**ECGF** is = $\triangle$**ABC**, and it has $\angle$**CEF** = $\angle$**D**.

PROPOSITION XII. PROBLEM.

To describe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given angle.



Let **ABCD** be the given rectilineal figure, and **E** the given ∠.

It is required to describe a ||^m *equal to* **ABCD,** *and having an* ∠ *equal to* ∠**E.**

Join **DB.**

Through **C** draw **CF** || to **DB,** to meet **AB** produced in **F.**

Join **DF.**

Then △**FDB** = △**CDB,** (II., 7.)

and to each adding △**DAB,**

△**DAF** = figure **ABCD.**

Describe the ||^m **GFHL** = △**DAF,** and having ∠**FGL** = given ∠**E.** (II., 11.)

Then ||^m **GFHL** is equal to the given figure **ABCD,** and it has the ∠**FGL** equal to the given ∠**E.**

In the same way, if the given figure have five or more sides, it can, by a repetition of the construction given in the Proposition, be reduced to a triangle, and a parallelogram may be described equal to it, with an angle equal to the given angle.

Exercises.

1. Construct a rectangle equal to the sum of two given rectangles.

2. Construct a rectangle equal to the difference of two given rectangles.

3. Construct a rectangle equal to the sum of two given squares.

4. Construct a triangle equal in area to the triangle ABC, but having one of its sides of given length BD. (Lay off BD on BC or BC produced.)

5. Construct a triangle equal in area to a given triangle and having a given altitude.

6. ABC is a given triangle, and D a given point. Construct a triangle equal in area to ABC and having its vertex at D.

7. Construct a triangle equal in area to a given quadrilateral ABCD, the triangle having a given altitude.

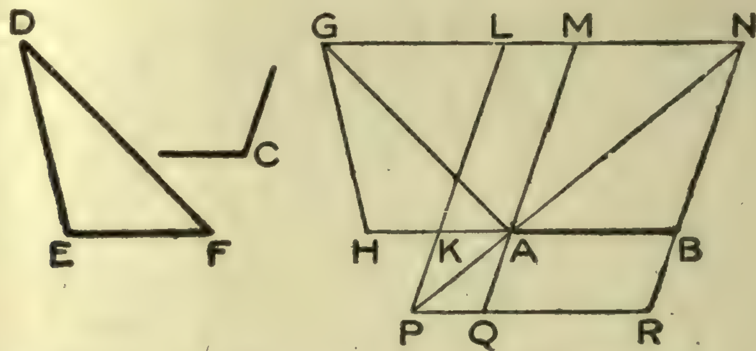
8. Show how to divide a triangle into n equal parts by straight lines drawn through one of its angles.

9. Hence show how to cut off from a given triangle an nth part by a straight line drawn through a point in one of its sides.

10. Construct a triangle equal in area to the triangle ABC, and having BD (in BC or BC produced) for one of its sides, and one of its angles of any given magnitude.

PROPOSITION XIII. PROBLEM.

On a given straight line to describe a parallelogram equal in area to a given triangle, and having an angle equal to a given angle.



Let **AB** be the given st. line, **DEF** the given △, and **C** the given ∠.

It is required to describe a $\|^m$ *on* **AB**, *equal to* **DEF**, *and having an* ∠ *equal to* **C**.

On **BA** produced construct a △**GHA** equal to △**DEF**. (I., 1.)

Bisect **AH** in **K**. (I., 10.)

At **A** in **KA** construct the ∠**KAM** = ∠**C**. (I., 5.)

Through **G** draw **GN** ∥ to **AB**. (II., 1.)

Through **K, B** draw **KL, BN** ∥ **AM**, and meeting **GN** in **L** and **N**, respectively.

Join **NA**, and produce it to meet **LK** produced in **P**.

Through **P** draw **PQR** ∥ to **AB**, and meeting **MA** produced in **Q**, and **NB** produced in **R**.

Then **AR** is the $\|^m$ required.

For since **AR** and **AL** are complements of the $\|^{ms}$ about the diameter **PN** of the $\|^m$ **LR**, they are equal to one another. (II., 10.)

But since **K** is the bisection of **AH**, the $\|^m$ **AL** is equal to the △**GHA** (II., 11), that is, to the △**DEF**;

∴ $\|^m$ **AR** = △**DEF**.

Also its $\angle$**QAB** is = $\angle$**KAM**, which is = $\angle$**C**.
$\therefore \|^m$ **AR** has been described on **AB**, = $\triangle$**DEF**, and with $\angle$**QAB** = $\angle$**C**.

NOTE. If, instead of a triangle, the given figure be any polygon, such polygon may be reduced to a triangle of equal area, and the construction may then proceed as in the Proposition.

Exercises.

1. On each of two sides of an equilateral triangle a parallelogram is described. Show how to apply to the third side a parallelogram whose area is equal to the sum of these two areas, and having an angle of given magnitude.

2. Construct a rectangle equal to a given square, when the length of one side of the rectangle is given.

3. Describe a triangle equal to a given parallelogram, having a side of given length and an angle of given magnitude.

4. Construct a parallelogram equal in area to a given triangle and having the same perimeter as the triangle.

5. Construct a rhombus equal in area to a given parallelogram, and having a side common with the parallelogram.

6. On one side of a quadrilateral construct a rectangle equal in area to the given quadrilateral.

7. On a given base make a rectangle equal to a given rectangle.

8. Construct a rhombus equal in area to a given triangle, and having its sides equal to a given straight line. When does the construction become impossible?

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9. Show that the sides of squares of equal area are equal.

10. Describe a triangle equal to a given parallelogram, and having an angle equal to a given rectilineal angle.

11. The sum of the two parallel sides of a trapezium is double the line joining the middle points of the other two sides.

12. Given the middle points of the sides of a triangle, construct the triangle.

13. If on the sides of a square, points be taken equidistant from each of the angles, taken in order, these points are the angular points of another square.

14. If in the preceding exercise the points be taken on the sides of a rhombus, and joined, what is the figure so formed?

15. ABCD is a parallelogram on a fixed base AB, and of constant area. Find the locus of the intersection of its diagonals.

16. If a quadrilateral has one pair of opposite sides parallel, and two opposite angles equal, it is a parallelogram.

17. If a quadrilateral has two opposite sides equal, and also two opposite angles equal, both being obtuse, then the figure is a parallelogram.

18. ABC is a triangle. Construct a triangle equal in area, having its vertex at a given point in BC, and its base in the same straight line as AB.

19. Construct a triangle, having given the points of bisection of two sides, and a point of trisection of the third.

20. The triangle which has two of its sides equal to the diagonals of a quadrilateral, and the angle between them equal to either of the angles between these diagonals, is equal to the quadrilateral in area.

21. Construct a triangle equal in area to a given triangle ABC, having its base in the same straight line as AB and of given length, and its vertex in a given line not parallel to AB.

22. The feet of the perpendiculars drawn from A upon the internal and external bisectors of the angles at B and C of the triangle ABC, lie on the straight line joining the middle points of AB and AC.

23. Having given two sides of a triangle and the median from their intersection to the middle point of the opposite side, construct the triangle.

24. Having given the three medians of a triangle, construct the triangle.

25. Show that if the line joining the middle points of two opposite sides of a quadrilateral bisect the quadrilateral, these opposite sides are parallel.

26. If two parallelograms have a common diagonal, the angular points through which this diagonal does not pass are the corners of a third parallelogram.

BOOK III.

RATIO AND PROPORTION.

INTRODUCTION.

LINES IN PROPORTION.

SIMILAR TRIANGLES.

RELATION BETWEEN SQUARES ON SIDES OF A TRIANGLE.

Introduction.

1. If two quantities, A and B, are of the *same kind*, the one may be compared with the other.

This comparison may consist in considering whether A is greater than, equal to, or less than B.

Usually, however, a more profitable comparison consists in considering how many times A contains B, or what part A is of B; or, supposing A and B expressed in terms of a common unit, in considering what multiple, part, or parts, the number representing A is of that representing B.

Thus, if A and B be lines containing a common unit of length m and n times respectively, we receive a very *workable* comparison of A and B from considering what multiple, part, or parts, m is of n.

2. **Ratio is the relation of two magnitudes of the same kind to one another in respect of quantity,** the comparison of the magnitudes being made by considering what *multiple, part or parts*, the one is of the other.

The idea of ratio is exceedingly fundamental. It is not easy, in considering magnitude, to get away from the idea. It is with us when we speak of bisecting a line or angle; or of dividing a line into a number of equal parts; or above, when we speak of expressing A in terms of a unit; or when we speak of a parallelogram being double of a triangle.

The ratio of A to B is represented by the form A : B, or by $m : n$, where m and n are numbers expressing A and B respectively, in terms of a common unit.

√ Since the fraction $\frac{m}{n}$ represents the *multiple, part or parts,* that m is of n, it expresses the ratio $m:n$, that is the ratio of A to B.

√ A and B, or m and n, are called the **terms** of the ratio. The first term is called the **antecedent**; the second term the **consequent.**

3. PROPORTION: **Four magnitudes are said to be in proportion, or to be proportionals, when the ratio of the first to the second is equal to the ratio of the third to the fourth.**

Thus, if A, B, C, D be the magnitudes

$$A : B = C : D.$$

A and D are called the **extremes**; B and C the **means.** D is said to be a **fourth proportional** to A, B and C.

If A and B, expressed in terms of a common unit, be represented by a and b; and C and D, expressed in terms of a common unit, be represented by c and d, we have

$$\frac{a}{b} = \frac{c}{d}.$$

4. The truth of the following equations between these symbols will be at once apparent as consequences of the above equation:

$$ad = bc,$$

$$\frac{a}{c} = \frac{b}{d},$$

$$\frac{a+b}{b} = \frac{c+d}{d},$$

$$\frac{a-b}{b} = \frac{c-d}{d},$$

$$\frac{a+b}{a-b}=\frac{c+d}{c-d},$$

$$\frac{a+c}{c}=\frac{b+d}{d},$$

&c.

Conversely, if any of these equations holds, then $a:b=c:d$, or $\mathrm{A}:\mathrm{B}=\mathrm{C}:\mathrm{D}$.

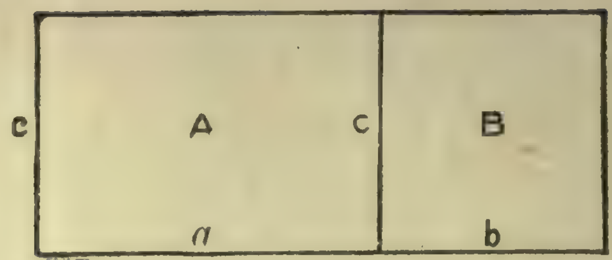
5. In dealing with ratio and proportion, we shall consider ourselves justified in passing at once from the letter-designations of magnitudes, to the numbers expressing the magnitudes in terms of known unit or units, and *vice versa.* Indeed, we may treat the letter-designations of lines, etc., as if they were numbers expressing the magnitudes of the lines, etc., and not only add and subtract but also multiply and divide them.

6. **Continued Proportion.**—Quantities are said to be in continued proportion when the ratio of the first to the second is equal to the ratio of the second to the third, and so on.

Thus, if $\mathrm{A}:\mathrm{B}=\mathrm{B}:\mathrm{C}=\mathrm{C}:\mathrm{D}=\ldots$; then A, B, C, D, . . are said to be in continued proportion.

If in the continued proportion three quantities only are involved, A, B and C,—so that $\frac{\mathrm{A}}{\mathrm{B}}=\frac{\mathrm{B}}{\mathrm{C}}$, or $\mathrm{AC}=\mathrm{B}^2$,—then C is said to be a **third proportional** to A and B; and B is said to be a **mean proportional** between A and C.

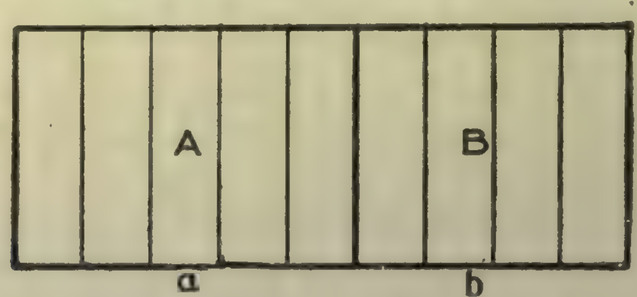
7. If there be two rectangles of the same altitude c, whose areas are represented by A and B, and bases by a and b, then $\mathrm{A}=ac$, $\mathrm{B}=bc$;



$$\therefore \frac{\mathbf{A}}{\mathbf{B}} = \frac{ac}{bc} = \frac{a}{b};$$

that is, **rectangles of the same altitude are to one another as their bases.**

[This result is so fundamental and so associated with our every-day experience, being the principle regulating the value of most articles that are sold by the length, that some may think even the above simple demonstration uncalled for. We must recollect, however, that the truth of the form $\mathbf{A} = ac$ depends on the fact that by drawing two sets of parallel lines, **A** can be divided into ac equal squares, and this in turn depends on principles that have been developed in connection with parallel lines and parallelograms. Or again, if we divide



the rectangles into equal parts by parallel lines drawn through the ends of each unit of length of base, and so obtain at once

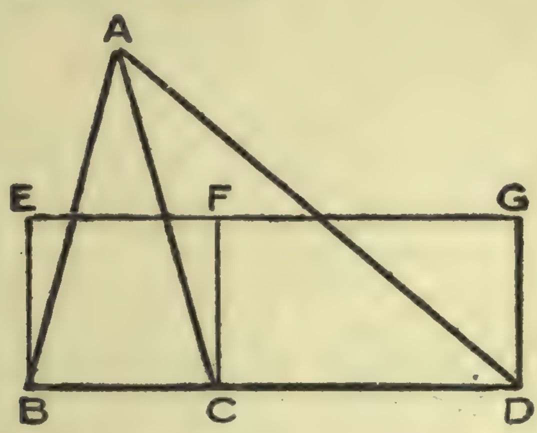
$$\mathbf{A} : \mathbf{B} = a : b,$$

we must remember that our assurance that these parts are equal comes from our knowledge of the properties of parallels and parallelograms.]

Lines in Proportion.

PROPOSITION I. THEOREM.

Triangles of the same altitude are to one another as their bases.



Let the $\triangle$s **ABC, ACD** have the same altitude, namely the $\perp$r from **A** on **BD**.

Then the $\triangle$**ABC** *is to the* $\triangle$**ACD** *as* **BC** *is to* **CD.**

On the bases **BC** and **CD**, construct the rectangles **BCFE, CDGF,** each of half the altitude of the $\triangle$s.

Then rect. **EC** = $\triangle$**ABC,** and rect. **FD** = $\triangle$**ACD.**

And $\triangle$**ABC** : $\triangle$**ACD** = rect. **EC** : rect. **FD,**

= **BC** : **CD.** (III., In., § 7.)

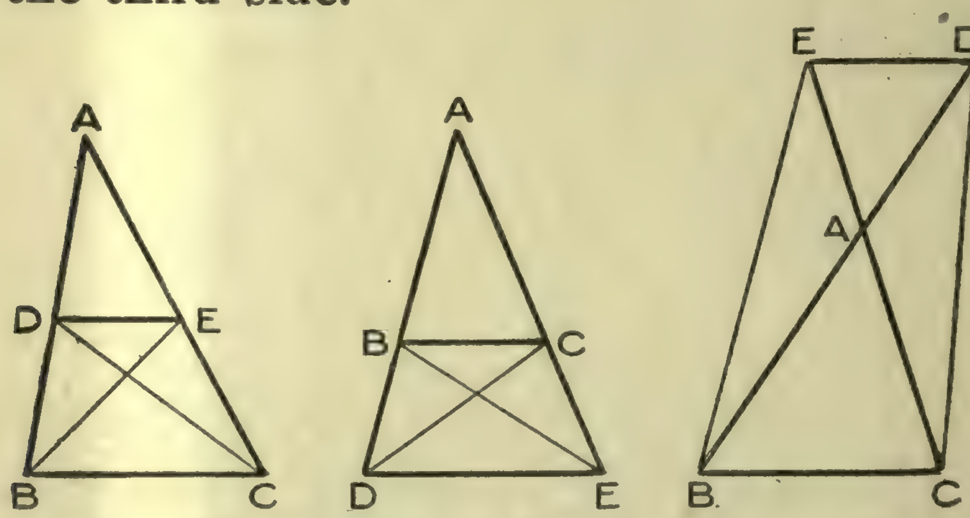
Proposition 9 of "Additional Propositions" after Book V., is an alternative proof of this theorem, and may be substituted for the preceding demonstration. It has, possibly, an advantage over the preceding proof, in showing clearly *why* the triangles are as their bases.

NOTE. In stating a proportion we may read the symbol : "is to," and = "as." Or we may say,—"The ratio of . . to . . is equal to the ratio of . . to . . ." We may use similar language where the ratios are expressed in the form of fractions.

PROPOSITION II. THEOREM.

If a straight line be drawn parallel to one side of a triangle, it cuts the other sides proportionally.

And conversely, if a straight line cut two sides of a triangle proportionally, it is parallel to the third side.



Let DE be || to BC, a side of the △ABC.

Then BD *is to* DA *as* CE *is to* EA.

Join BE, CD.

Because BC is || to DE; ∴ △EDB = △DEC.

And EDA is another △.

∴ △EDB : △EDA = △DEC : △EDA.

But △EDB : △EDA = BD : DA, (III., 1.)

and △DEC : △EDA = CE : EA;

∴ BD : DA = CE : EA.

Conversely, let BD be to DA as CE to EA.

Then DE *is* || *to* BC.

The same construction being made

since BD : DA = CE : EA;

and BD : DA = △EDB : △EDA, (III., 1.)

also CE : EA = △DEC : △EDA;

$$\therefore \triangle EDB : \triangle EDA = \triangle DEC : \triangle EDA;$$

and $\therefore \triangle EDB = \triangle DEC$.

And these $\triangle$s are on the same base DE;

$\therefore$ DE is $\parallel$ to BC. (II., 8.)

COROLLARY. Since

$$\frac{BD}{DA} = \frac{CE}{EA};$$

$$\therefore \frac{AD}{DB} = \frac{AE}{EC}.$$

Again, since

$$\frac{BD}{DA} = \frac{CE}{EA};$$

$$\therefore \frac{BD + DA}{DA} = \frac{CE + EA}{EA},$$

$$\text{or } \frac{AB}{AD} = \frac{AC}{AE}.$$

The same results may be obtained from the figures, by following the method of the Proposition.

An indirect proof of the converse may easily be arranged,—Draw DF parallel to BC, meeting AC in F, and then show that F and E coincide.

Proposition 10 of "Additional Propositions" after Book V., is an alternative proof of this theorem, and may be substituted for the preceding demonstration. It has, possibly, an advantage over the preceding proof, in showing clearly *why* the sides are cut proportionally.

Exercises.

1. The straight line joining the middle points of the sides of a triangle is parallel to the base, and is one-half the base.

2. If in a quadrilateral ABCD, the sides AB, CD be parallel, prove that any straight line parallel to AB divides AD and BC proportionally.

3. A fixed point O is joined to various points in a given straight line AB. Prove that all the points which divide the joining lines in the same ratio, lie on a straight line which is parallel to AB. Express this theorem as a question in loci.

4. Show that the triangles into which a quadrilateral is divided by its diagonals, form a proportion.

5. If ABCD be a trapezium having AB parallel to DC, and AC, BD intersect in O; show that the triangle OAD is a mean proportional between the triangles OCD, OAB.

6. The diagonals of a trapezium cut one another proportionally; and any straight line drawn parallel to either of the parallel sides cuts the other sides in the same ratio. Take before No 5.

7. The diagonals of a trapezium one of whose parallel sides is double the other, cut one another at points of trisection. Use No 1

8. The medians of a triangle cut one another at a point of trisection. Proof similar to No 7

9. ABCD is a parallelogram, and E, F are the middle points of the sides AD, BC, respectively. Show that BE, FD trisect AC.

10. In the same parallelogram, if G be the bisection of CD, show that BE, BG trisect AC. Use No 8 or No 7

11. From any point O on the diagonal AC of a quadrilateral ABCD, OP is drawn parallel to AB to meet BC in P, and OQ is drawn parallel to AD to meet CD in Q. Show that PQ is parallel to BD.

12. The point D divides the base BC of the triangle ABC in the ratio $m:n$. If O be any point in AD, show that the ratio of the areas OAB, OAC is $m:n$. Prove converse.

13. Find a point O within the triangle ABC, such that triangle OBC = 2 triangle OCA = 4 triangle OAB.

14. In a triangle ABC, the lines AD, BE, CF, drawn to the opposite sides intersect in O. If AF : FB $= p:q$ and BD : DC $= q:r$, and the area of the triangle OAF be represented by pk, obtain the following expressions for areas of triangles in the figure,—

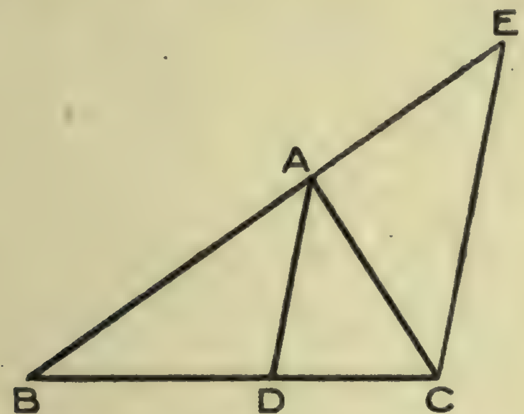
$$\text{OFB} = qk,\ \text{OAB} = (p+q)k,\ \text{OAC} = \frac{r}{q}(p+q)k,\ \text{BOC} = \frac{r}{p}(p+q)k.$$

Hence show that CE : EA $= r:p$.

Obtain expressions for areas of remaining triangles in the figure.

PROPOSITION III. THEOREM.

If the vertical angle of a triangle be bisected by a straight line which also cuts the base, the segments of the base have the same ratio which the other sides of the triangle have to one another.



Let the vertical $\angle$**BAC** of the $\triangle$**ABC** be bisected by **AD** which cuts the base in **D**.

Then **BD** *is to* **DC** *as* **BA** *to* **AC**.

Through **C** draw **CE** $\parallel$ to **AD**, to meet **BA** produced in **E**.

Then since **AD** is $\parallel$ to **CE**,

$$\angle ACE = \angle CAD;$$

$$\text{also } \angle AEC = \angle BAD;$$

$$\text{but } \angle BAD = \angle CAD; \qquad \text{(Hyp.)}$$

$$\therefore \angle AEC = \angle ACE,$$

$$\text{and } AE = AC.$$

But **BD** : **DC** = **BA** : **AE**, since **AD** is $\parallel$ to **EC**; (III., 2.)

$$\therefore BD : DC = BA : AC.$$

Conversely, if the segments of the base have the same ratio which the sides of the triangle have to one another, the straight line drawn from the vertex to the point of section bisects the vertical angle.

Let **BD** be to **DC** as **BA** to **AC**.

Then **AD** *bisects the* $\angle$**BAC**.

The same construction being made,

since **BD** : **DC** = **BA** : **AC**, (Hyp.)

and **BD** : **DC** = **BA** : **AE**, **AD** being $\parallel$ to **EC**;

$\therefore$ **AE** = **AC**,

and $\angle$**AEC** = $\angle$**ACE**.

But $\angle$**AEC** = $\angle$**BAD**,

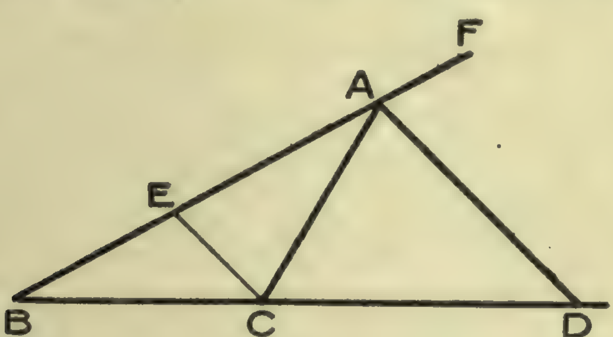
and $\angle$**ACE** = $\angle$**DAC**;

$\therefore$ $\angle$**BAD** = $\angle$**DAC**,

and **AD** bisects $\angle$**BAC**.

PROPOSITION IV. THEOREM.

If the exterior angle at the vertex of a triangle be bisected by a straight line which also cuts the base produced, the segments of the base have the same ratio which the other sides of the triangle have to one another.



Let the exterior $\angle$ at the vertex **A** of the $\triangle$**ABC** be bisected by **AD**, which cuts the base in **D**.

Then **BD** *is to* **DC** *as* **BA** *to* **AC.**

Through **C** draw **CE** $\parallel$ to **AD**, to meet **BA** in **E.**

Then, since **AD** is $\parallel$ to **CE**,

$\angle$**ACE** = $\angle$**CAD**;

also $\angle$**AEC** = $\angle$**FAD**;

but $\angle$**FAD** = $\angle$**CAD**; (Hyp.)

$\therefore$ $\angle$**ACE** = $\angle$**AEC**,

and **AE** = **AC.**

But **BD** : **DC** = **BA** : **AE**, since **AD** is $\parallel$ to **EC**; (III., 2.)

$\therefore$ **BD** : **DC** = **BA** : **AC.**

Conversely, if the segments into which the base is divided externally have the same ratio which the sides of the triangle have to one another, the straight line drawn from the vertex to the point of section bisects the exterior angle at the vertex.

Let **BD** be to **DC** as **BA** to **AC**.

Then **AD** *bisects the exterior* $\angle$ at **A**.

The same construction being made,

since **BD : DC = BA : AC**, (Hyp.)

and **BD : DC = BA : AE**, **AD** being $\parallel$ to **EC**;

$\therefore$ **AE = AC**,

and $\angle$ **AEC** = $\angle$ **ACE**.

But $\angle$ **AEC** = $\angle$ **FAD**,

and $\angle$ **ACE** = $\angle$ **DAC**;

$\therefore$ $\angle$ **FAD** = $\angle$ **DAC**,

and **AD** bisects $\angle$ **CAF**.

Propositions III. and IV. are in reality one and the same proposition. An enunciation applicable to both may be given as follows,—*If a straight line bisect either of the angles which two straight lines make with one another, and a transversal fall across the three, the segments of the transversal between the bisecting line and the two straight lines, have the same ratio as the intercepts made by the transversal on the two straight lines.* It is to be noted that the word "triangle" is not mentioned in the preceding statement.

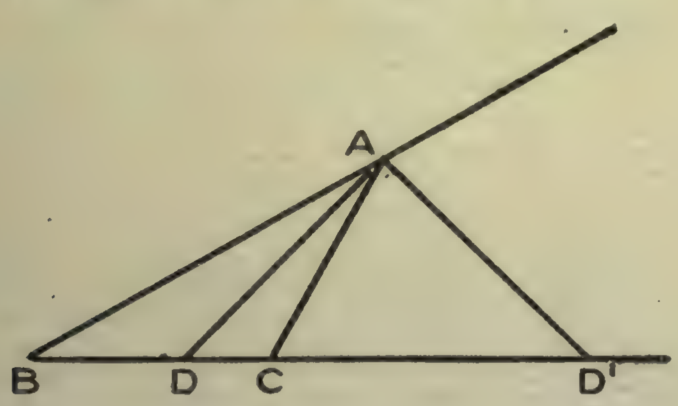
An analogous enunciation may be formed to include both converses.

NOTE. The term *transversal* is applied to a line falling across a number of other lines, especially when these latter lines radiate from a point. Lines radiating from a point are said to form a *pencil of rays.*

In Prop. III., **BC** is divided *internally* at **D**; in Prop. IV., *externally* at **D**; and in the latter case **D** is considered as much a point of division of **BC**, and **BD**, **DC** as much segments of **BC**, as in the former case.

Brevity might have been secured by combining Props. III. and IV. into one, but it is at least no disadvantage to the learner to repeat in IV. the reasoning he has already met with in III.

If both internal and external angles at the vertex are bisected, we obtain the following result,—



$$\frac{BD}{DC}=\frac{BA}{AC}=\frac{BD'}{D'C};$$

$$\therefore \frac{DC}{BD}=\frac{D'C}{BD'};$$

$$\therefore \frac{BC-BD}{BD}=\frac{BD'-BC}{BD'};$$

$$\therefore \frac{BC}{BD}-1=1-\frac{BC}{BD'};$$

$$\therefore \frac{1}{BD}-\frac{1}{BC}=\frac{1}{BC}-\frac{1}{BD'};$$

Hence, $\frac{1}{\text{BD}}, \frac{1}{\text{BC}}, \frac{1}{\text{BD}'}$ are in Arithmetic Progression, and therefore **BD, BC, BD′** are in Harmonic Progression; and the straight line **BC** is said to be *harmonically* divided.

Generally a straight line is harmonically divided when it is divided internally and externally in the same ratio. **BC** is thus divided.

The points **B, D, C, D′** are said to form a *harmonic range;* and the lines **AB, AD, AC, AD′** a *harmonic pencil.*

Exercises.

√ 1. A point D is taken in BC, a side of the triangle ABC, and the angles ADB, ADC are bisected by DE, DF, meeting AB, AC in E and F, respectively. If EF be parallel to BC, show that D must be the bisection of BC.

√ Prove also the converse of this,—that if D be the middle point of BC, then EF is parallel to BC.

2. If a, b, c be the sides of the triangle ABC, opposite the angles A, B, C, and D, D′ the points where the internal and external bisectors of A meet BC, prove that

$$\text{DD}' = \frac{2abc}{b^2 - c^2}, \text{ or } \frac{2abc}{c^2 - b^2}.$$

3. If BC be divided harmonically in D and D′, show that

$$\frac{1}{\text{BD}} + \frac{1}{\text{BD}'} = \frac{2}{\text{BC}}.$$

4. AD bisects the angle A of the triangle ABC, and meets the base in D. E is the middle point of BC. Show that

$$\text{ED} = \frac{a\,(b-c)}{2\,(b+c)}, \text{ or } \frac{a\,(c-b)}{2\,(b+c)},$$

where a, b, c are the sides opposite to A, B, C.

Similar Triangles.

Two rectilineal figures are said to be equiangular when the angles of the first, taken in order, are equal respectively to the angles of the second, taken in the same order.

In such figures, a side of one is said to **correspond** to a side of the other, when the angles adjacent to the former side correspond to the angles adjacent to the latter side. Such sides are called **corresponding sides.**

Two rectilineal figures are said to be similar when they are equiangular, and the ratio of each side of the one to the corresponding side of the other is the same.

That two figures may be similar it is necessary that both conditions be satisfied; *i.e.*,—

(1) The figures must be equiangular.

(2) The ratio between corresponding sides must be constant.

Two figures may fulfil the second condition, *e.g.*, a square and a rhombus, without fulfilling the first; and two figures may satisfy the first condition, *e.g.*, a square and a rectangle, without satisfying the second.

It will be shown, however, that in the particular case of triangles each condition involves the other.

Figures which are similar are, in popular language, spoken of as having the same *shape*, though possibly differing in size.

In stating the proportionality of the sides of two triangles ABC, DEF, the form

$$\mathrm{AB : DE = BC : EF = CA : FD},$$

$$\text{or } \frac{\mathrm{AB}}{\mathrm{DE}} = \frac{\mathrm{BC}}{\mathrm{EF}} = \frac{\mathrm{CA}}{\mathrm{FD}}$$

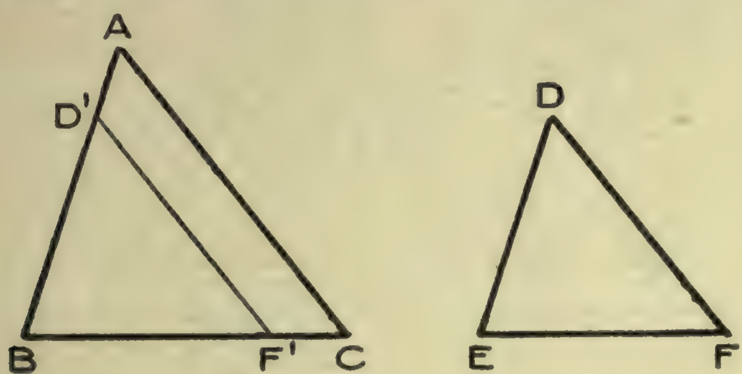
is preferable to

$$\frac{\mathrm{AB}}{\mathrm{BC}} = \frac{\mathrm{DE}}{\mathrm{EF}}, \quad \frac{\mathrm{BC}}{\mathrm{CA}} = \frac{\mathrm{EF}}{\mathrm{FD}}, \quad \frac{\mathrm{CA}}{\mathrm{AB}} = \frac{\mathrm{FD}}{\mathrm{DE}},$$

since the former statement is briefer, and makes prominent the constancy of the ratio between corresponding sides.

PROPOSITION V. THEOREM.

If two triangles are equiangular, the ratios of corresponding sides are equal.



Let **ABC**, **DEF** be two Δs, in which $\angle$**A** $=$ $\angle$**D**, $\angle$**B** $=$ $\angle$**E** and $\therefore$ $\angle$**C** $=$ $\angle$**F**.

Then **AB** *is to* **DE** *as* **BC** *to* **EF**, *as* **CA** *to* **FD**.

Let the Δ**DEF** be placed on Δ**ABC**, so that **E** coincides with **B**, **ED** falls on **BA**, and **EF** on **BC**.

Let **D′**, **F′** be the new positions of **D**, **F**.

Then since $\angle$**D′** $=$ $\angle$**A**, **D′F′** is $\parallel$ to **AC**;

$$\therefore \mathbf{AB} : \mathbf{D'B} = \mathbf{BC} : \mathbf{BF'};$$

$$i.e., \ \mathbf{AB} : \mathbf{DE} = \mathbf{BC} : \mathbf{EF}.$$

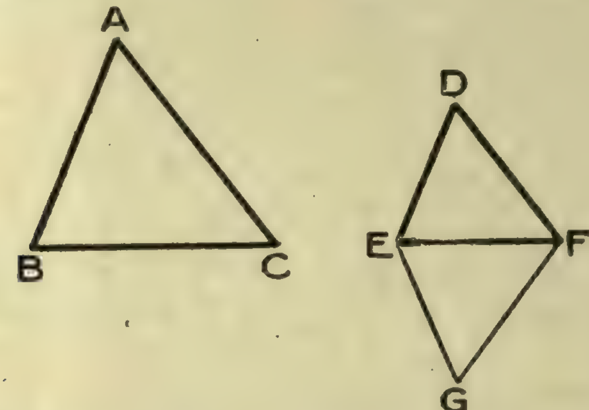
Similarly, by placing the Δ**DEF** on Δ**ABC** so that **F** coincides with **C**, **FE** falls on **CB** and **FD** on **CA**, it may be proved that **BC** : **EF** $=$ **CA** : **FD**.

$$\therefore \frac{\mathbf{AB}}{\mathbf{DE}} = \frac{\mathbf{BC}}{\mathbf{EF}} = \frac{\mathbf{CA}}{\mathbf{FD}}.$$

Hence Δs which are equiangular have the ratios between corresponding sides equal, and are therefore *similar*. (p. 121.)

PROPOSITION VI. THEOREM.

If the ratios of the three sides of one triangle to the three sides of another triangle, taken in order, be equal, the triangles are equiangular.



Let **ABC, DEF** be two Δs, in which **AB : DE = BC : EF = CA : FD.**

Then the Δs *are equiangular.*

On the side of **EF** remote from **D,** construct the $\angle$**GEF** = $\angle$**B,** and the $\angle$**GFE** = $\angle$**C.**

Then Δs **ABC, GEF** are equiangular;

$\therefore$ **AB : GE = BC : EF.** (III., 5.)

But **AB : DE = BC : EF ;** (Hyp.)

$\therefore$ **GE = DE.**

Similarly it may be proved that **GF = DF.**

Hence the Δs **DEF, GEF** are equiangular.

But Δs **ABC, GEF** are equiangular;

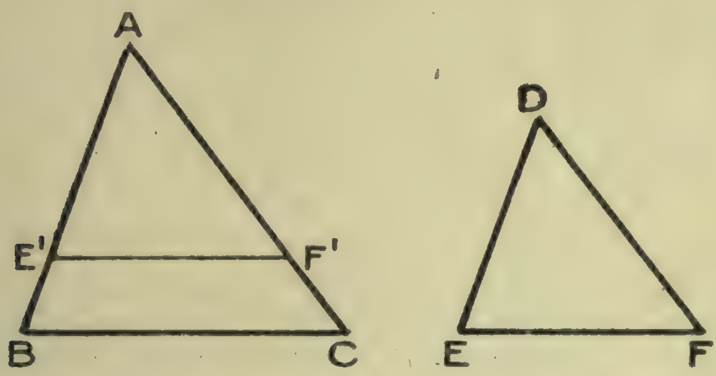
$\therefore$ Δs **ABC, DEF** are equiangular.

Triangles, therefore, which have the ratios of the three sides of the one to the three sides of the other, taken in order, equal, are *similar.*

It will be noted that it is between corresponding sides, according to the definition of corresponding sides, that the ratio is constant.

PROPOSITION VII. THEOREM.

If two triangles have one angle of the one equal to one angle of the other, and the sides about the equal angles proportionals, the triangles are equiangular.



Let **ABC, DEF** be two Δs, in which $\angle$**A** = $\angle$**D**, and **AB** is to **DE** as **AC** to **DF**.

Then the Δs *are equiangular.*

Let the Δ**DEF** be placed on the Δ**ABC**, so that **D** falls on **A, DE** on **AB**, and **DF** on **AC**; and let **E′, F′** be the new positions of **E** and **F.**

Then since **AB : DE = AC : DF,**

i.e., **AB : AE′ = AC : AF′**;

$\therefore$ **E′F′** is $\parallel$ to **BC.** (III., 2.)

Hence $\angle$**B** = $\angle$**E′** = $\angle$**E,**

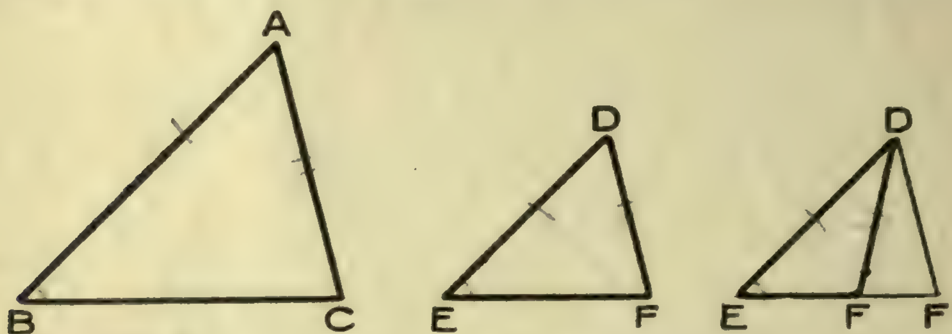
and $\angle$**C** = $\angle$**F′** = $\angle$**F**;

and the triangles are equiangular.

Therefore, if two triangles have an angle in each equal, and the sides about these angles proportionals, the triangles are *similar.*

PROPOSITION VIII. THEOREM.

If the ratios of two sides of one triangle to two sides of another triangle be equal, and if the angles opposite to one pair of these sides be equal, the angles opposite to the other pair of sides are either equal or supplementary.



Let **ABC, DEF** be two $\triangle$s, in which **AB** is to **DE** as **AC** to **DF**, and $\angle \mathbf{B} = \angle \mathbf{E}$.

Then either the $\angle$s *at* **C** *and* **F** *are equal, and* $\therefore$ *the* $\triangle$s *equiangular; or* $\angle \mathbf{C} + \angle \mathbf{F} = 2$ *rt.* $\angle$s.

If $\angle \mathbf{C}$ be not $= \angle \mathbf{F}$, then $\angle \mathbf{A}$ is not $= \angle \mathbf{D}$.

Make then $\angle \mathbf{EDF'} = \angle \mathbf{BAC}$, and produce **EF**, if necessary, to meet **DF′** in **F′**.

Then $\triangle$s **ABC, DEF′** are equiangular;

$\therefore$ **AB** : **DE** = **AC** : **DF′**. (III., 5.)

But **AB** : **DE** = **AC** : **DF**; (Hyp.)

$\therefore$ **DF** = **DF′**;

and $\therefore$ $\angle \mathbf{DF'F} = \angle \mathbf{DFF'}$.

Hence $\angle \mathbf{C} = \angle \mathbf{DF'F} = \angle \mathbf{DFF'} = 2$ rt. $\angle$s $- \angle \mathbf{DFE}$;

and $\angle \mathbf{C} + \angle \mathbf{F} = 2$ rt. $\angle$s.

Therefore, if the angles at **C** and **F** be not supplementary, the triangles are equiangular and therefore *similar.*

It may be that something in the data respecting the triangles will show that the angles at **C** and **F** cannot be supplementary, *i.e.*, that the triangles are similar.

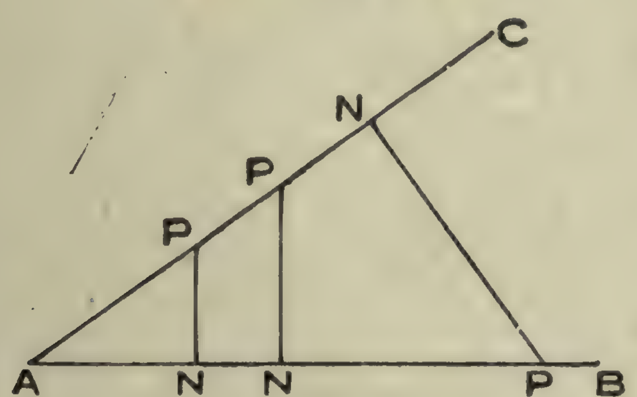
Angles which are supplementary are both right angles, or one is acute and the other obtuse.

Hence, if the given angles at **B** and **E** be right angles, or obtuse angles, the angles at **C** and **F** must be both acute, and cannot be supplementary; and the triangles are similar.

Again, if **AC** be greater than, or equal to **AB** (and therefore **DF** greater than, or equal to **DE**), the angle **ABC** is greater than or equal to the angle **ACB**, and the angle **DEF** greater than or equal to the angle **DFE**. Hence both the angles **ACB, DFE** are acute; and the triangles are similar.

Lastly, if for any reason we know the angles **ACB, DFE** to be both acute, or both obtuse, they cannot be supplementary; and the triangles are similar. If one of the angles **ACB, DFE** be known to be a right angle, the other is either equal to it, or supplementary, *i.e.*, again equal to it; and the triangles are similar.

TRIGONOMETRICAL RATIOS. If **BAC** be any angle, and if from any point **P** in **AC** or **AB**, a perpendicular **PN**



be drawn to **AB** or **AC**, the following names are given to the ratios of the sides of the right-angled triangle **PAN**, to one another:

The ratio $\frac{PN}{AP}$ is called the *sine* of the $\angle A$, written sin **A**.

"	$\frac{AN}{AP}$	"	*cosine*	" , "	cos **A**.
"	$\frac{PN}{AN}$	"	*tangent*	" , "	tan **A**.
"	$\frac{AN}{PN}$	"	*cotangent*	" , "	cot **A**.
"	$\frac{AP}{AN}$	"	*secant*	" , "	sec **A**.
"	$\frac{AP}{PN}$	"	*cosecant*	" , "	cosec **A**.

These six ratios are called the **trigonometrical ratios** of the $\angle A$.

It will be observed that though three different triangles, **PAN**, are constructed, they are all similar to one another, and each of the above ratios has the same value for all of them. In other words *each of the above ratios is constant for the same angle.*

Construct angles of 29°, 35°, 43°, 57°, 69°, 75°; for each accurately form the right-angled triangle **PAN**; measure accurately the lengths of its sides to the nearest millimetre; and calculate to two decimal places the numerical values of the six trigonometrical ratios for each of these angles.

The best results will be obtained by making the triangle **PAN** somewhat large,—say with a base **AN** of about 100 millimetres.

Exercises.

1. If ABC, DEF be similar triangles, and the ratio between corresponding sides be represented by k, show that any two corresponding lines in the two figures are in this ratio k; for example,—

(1) **Perpendiculars from corresponding angles on opposite sides are in the ratio k.**

(2) **Corresponding segments between feet of perpendiculars and corresponding angles, are in the ratio k.**

(3) **Medians from corresponding angles are in the ratio k.**

(4) **Lines bisecting corresponding angles and terminated by opposite sides are in the ratio k.**

(5) **Lines from the bisections of corresponding sides to the corresponding trisections of corresponding sides are in the ratio k.**

(6) **Corresponding segments of sides made by the lines in (3) and (4) are in the ratio k.**

2. If any straight line EF, parallel to the side BC of a triangle ABC, cut AB, AC in E and F, show that EF is bisected by the median from A.

3. AOB is a perpendicular to two parallel lines, and COD, terminated by the parallel lines, AC, DB, passes through any point O in AOB. Show that AO : OB = CO : OD.

4. ABC, DBC are any two triangles on the same base BC, and between the same parallels. Show that the parts intercepted by the sides of the triangles on any straight line parallel to BC are equal.

5. Every straight line parallel to the base of a triangle cuts off a similar triangle.

6. ABC is a triangle having the angle B double of the angle C, and the bisector of B meets AC in D. Show that the triangles ABC, ADB are similar, and that AB is a mean proportional between AC and AD.

7. Show that two isosceles triangles are similar, if their vertical angles are equal.

8. Show that the ratio of the perpendiculars from two given points A and B, on any straight line which cuts AB in a fixed point, is constant.

9. ABCD is a quadrilateral in which AB is parallel to and half of CD. Show that any line through O, the intersection of the diagonals, and terminated by AB, CD, is trisected at O.

10. ABCD is a quadrilateral having the sides AB, CD parallel. Show that the line joining the middle point of CD to the intersection of the diagonals, will, on being produced, bisect AB.

11. ABC is a triangle, and perpendiculars AD, BE are drawn to the opposite sides, intersecting in P. Show that AP.PD = BP.PE.

12. Three lines AB, AC, AD pass through a point, and parallel lines are drawn intersecting them. Show that the segments into which AC divides the portions of the parallels intercepted by AB, AD, are in a constant ratio for a given direction of the parallels.

13. AB and CD are two parallel lines, and CD is bisected at E. AC, BE meet in F, and AE, BD meet in G. Show that FG is parallel to AB.

14. AB, CD are two parallel lines, and AD, BC intersect in O. Show that the triangles AOB, AOC, COD form a continued proportion.

15. In the figure of the preceding question

$$\frac{\triangle AOB}{\triangle AOC} = \frac{BO}{OC} = \frac{AB}{CD},$$

$$\text{also } \frac{\triangle AOC}{\triangle COD} = \frac{AO}{OD} = \frac{AB}{CD}.$$

What therefore is the ratio of the areas of the triangles AOB, COD? What sort of triangles are these?

If two triangles ABC, DEF be similar, place them so as to form a figure analogous to that in the preceding question, and hence show that their areas are as BC^2 to EF^2.

16. Through any point D in the base BC of the triangle ABC, DE and DF are drawn parallel to AB, AC, meeting AB, AC in F and E. Show that the triangle AFE is a mean proportional between the triangles DCE, DBF.

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17. From a figure calculate the cosine and tangent of 27°, and apply your results to find AB and BC in the triangle ABC, where

$$A = 27°, C = 90°, AC = 230 \text{ feet}.$$

18. From a figure calculate the cosine of 70°, and apply your result to the solution of the following problem: To find the distance of an object B from A, a base line AC is measured and found to be 750 feet; the angles A and C are measured and found to be 70° and 90° respectively. Express AB in feet.

19. From a figure calculate the sine of 73°, and apply your result to the solution of the following problem: In a triangle ABC, the sides AB, BC are 720 and 480 feet respectively, and the $\angle B$ is 73°; find the length of the perpendicular from A on BC, and calculate the area of the triangle.

20. In a triangle ABC, if b and c be the lengths of the sides opposite the angles B and C respectively, show that the area of the triangle is $\frac{1}{2}bc \sin A$.

Exercises such as the following are of great practical value:

21. P is a distant object. A base line AB of 720 feet is measured, and the angles PAB, PBA are found to be 75° and 80° respectively. By constructing a triangle P′A′B′ equiangular to PAB, having A′B′ 50 millimetres in length, and measuring P′A′, P′B′ in millimetres, use the property of similar triangles to find approximately PA and PB.

22. Similarly, AB is 200 yards; angles PAB, PBA are 80° and 85° respectively; A′B′ is 80 millimetres. Find approximately PA and PB by measuring the lengths of P′A′, P′B′ in millimetres.

23. Similarly, AB is 480 feet; the angles PAB, PBA are 63° and 70° respectively; A′B′ is 55 millimetres. Find approximately PA and PB by measuring the lengths of P′A′, P′B′ in millimetres.

24. In the preceding question, test the accuracy of your result by constructing P′A′B′ with B′A′=100 millimetres, or A′B′=75 millimetres, measuring P′A′, P′B′ in millimetres, and thence again, by similar triangles, finding PA and PB.

25. The triangle ABC has $\angle$A=35°, AB=270 ft., AC=480 ft. Construct a triangle A′B′C′ with $\angle$A′=35°, A′B′=2¼ in., A′C′=4 in., which thus is similar to ABC. Employ the similarity of the triangles to find approximately $\angle$B, $\angle$C and side BC.

26. The triangle ABC has $\angle$A=57°, AB=340 yds., AC=460 yds. Construct a triangle A′B′C′ with $\angle$A′=57°, A′B′=68, A′C′=92 millimetres, which thus is similar to ABC. Employ the similarity of the triangles to find approximately $\angle$B, $\angle$C and side BC.

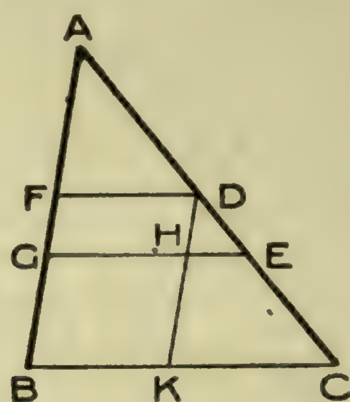
27. The triangle ABC has $\angle$B=75°, AB=750 ft., AC=1100 ft. Construct on paper a similar triangle, and use the similarity of the triangles to find approximately $\angle$A, $\angle$C and side BC.

In such numerical exercises as the seven preceding, the teacher should solve the triangles by the usual trigonometrical formulæ, that he may inform the class as to the closeness of their approximations reached by instrumental methods.

These numerical exercises may, of course, be multiplied indefinitely.

PROPOSITION IX. PROBLEM.

To divide a given straight line similarly to a given divided straight line.



Let **AB** be the given st. line to be divided, and **AC** the other given st. line divided at **D** and **E**.

It is required to divide **AB** *similarly to* **AC**.

Let **AB, AC** be placed so as to be coterminous at **A**.

Join **BC**.

Through **D** and **E** draw **DF, EG** ∥ to **BC**.

Then **AB** is divided at **F, G** similarly to **AC**.

Through **D** draw **DHK** ∥ to **AB**.

Since **HE** is ∥ to **KC**;

∴ **DH : HK = DE : EC**. (III., 2.)

But **DH = FG**, and **HK = GB**; (II., 5.)

∴ **FG : GB = DE : EC**.

Again, since **FD** is ∥ to **GE**;

∴ **AF : FG = AD : DE**. (III., 2.)

Hence **AB** has been divided at **F** and **G** similarly to **AC** at **D** and **E**.

COROLLARY. If it be required to divide a given line in a given ratio, as the ratio **AB** : **BC**, let **AB**, **BC** be

placed so as to be in a straight line, and divide the given line similarly to **AC** by the method of the proposition.

NOTE. When **AB** has been divided similarly to **AC**, whatever relation exists between the segments of **AC**, the same relation exists between the corresponding segments of **AB**. Thus, if **A, D, E, C** form a harmonic range, so also do **A, F, G, B**. If **AE** is divided at **D** so that the rectangle **AE.ED** equals the square on **AD**, then also $\mathbf{AG.GF} = \mathbf{AF}^2$; and so on. This proposition is most easily seen in its general form, as follows: Let a, b, c, ... be the segments of **AC**, and a', b', c', ... the corresponding segments of **AB**. Then

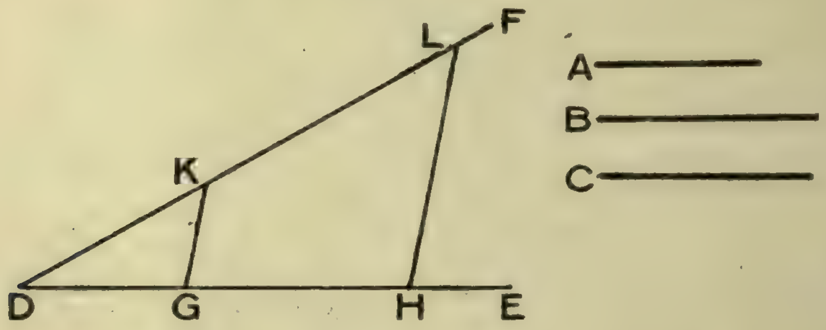
$$\frac{a}{a'} = \frac{b}{b'} = \, .. = k, \text{ say };$$

$$\text{so that } a = a'k,\ b = b'k,\ \ldots$$

The relation between a, b, .. will be expressed by an equation which must be homogeneous, for we could not have an expression in which lines would be added to rectangles, and both to cubes. Substituting, then, $a'k$, $b'k$, .. for a, b, .. , we shall find that k divides out, and shall have the original equation, but with a', b', .. replacing a, b, ...

PROPOSITION X. PROBLEM.

To find a fourth proportional to three given straight lines.



Let **A, B, C** be the three given st. lines.

It is required to find a fourth proportional to them.

Draw the lines **DE, DF.**

Take **DG** = **A, GH** = **B,** and **DK** = **C.**

Join **GK,** and draw **HL** ∥ to **GK.**

Then **KL** is a fourth proportional to **A, B, C.**

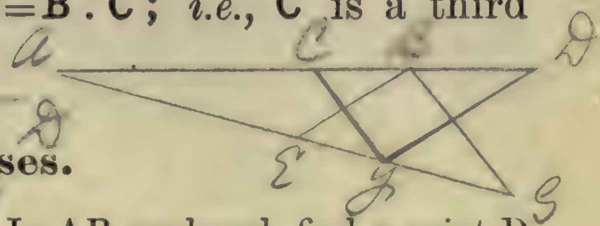
Since **GK** is ∥ to **HL**;

∴ **DG : GH = DK : KL.** (III., 2.)

Hence **A : B = C : KL,**

and **KL** is a fourth proportional to **A, B, C.**

COROLLARY. The proposition includes the finding of a third proportional to two given lines, **A, B.** For repeat the line **B,** and find a fourth proportional to **A, B, B.** Let this be **C.** Then **A : B = B : C**; *i.e.*, **C** is a third proportional to **A, B.**



Exercises.

1. C is any point in the line AB. In AB produced, find a point D such that

$$\frac{AD}{DB} = \frac{AC}{CB}.$$

2. If A be a point without a circle, whose centre is O, and lines drawn from A to the circumference be divided in a constant ratio, show that the locus of the point of division is a circle whose centre is the point where AO is divided in the same ratio.

3. Given the sum of two lines and their ratio; find the lines.

4. Given the difference of two lines and their ratio; find the lines.

5. Find the locus of a point which moves so that the perpendiculars from it, on two given straight lines, are in a given ratio.

6. Find a point O within a triangle ABC, such that if perpendiculars be dropped on the sides a, b, c, the ratios of the perpendiculars may be $a : b : c$.

7. Find three lines which are to one another as three given rectangles are to one another. (Prop. ~~10~~ 13, Bk. II.)

8. Divide a triangle ABC into three triangles OBC, OCA, OAB, such that these triangles may be in given ratios.

9. OX, OY are two given straight lines, and OA, OB are taken on them respectively, so that the sum of OA, OB is constant. Lines are drawn through A and B parallel to OY, OX; show that the locus of their intersection is a straight line.

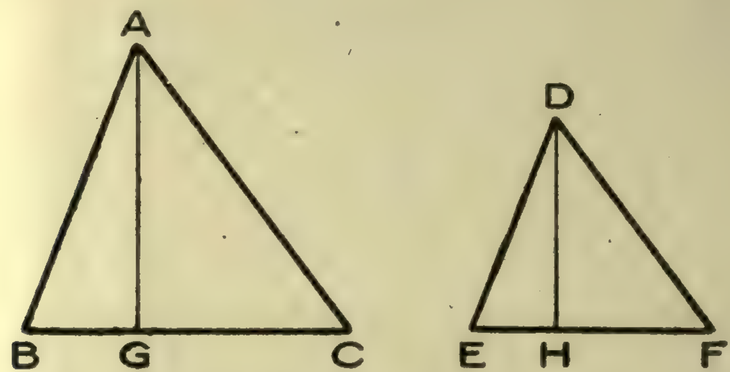
10. ABCD is a parallelogram, and DEF is drawn cutting AB in E and CB produced in F. Show that CF is a fourth proportional to EA, AD, AB.

11. If the perpendiculars from two fixed points on a straight line passing between them be in a given ratio, the straight line must pass through a third fixed point.

12. Find a straight line such that the perpendiculars on it from three fixed points may be in given ratios to each other.

PROPOSITION XI. THEOREM.

If two triangles are similar, the ratio of their areas is equal to the ratio of the squares on their corresponding sides.



Let **ABC**, **DEF** be similar $\triangle$s.

The ratio of their areas is equal to the ratio of the squares on corresponding sides.

Let **AG**, **DH** be the altitudes of the $\triangle$s.

$$\text{Then } \frac{\triangle ABC}{\triangle DEF} = \frac{\frac{1}{2}BC \cdot AG}{\frac{1}{2}EF \cdot DH} = \frac{BC}{EF} \cdot \frac{AG}{DH}.$$

But since $\angle B = \angle E$, and $\angle G = \angle H$, the $\triangle$s **ABG**, **DEH** are similar;

$$\therefore \frac{AG}{DH} = \frac{AB}{DE}.$$

$$\therefore \frac{\triangle ABC}{\triangle DEF} = \frac{BC}{EF} \cdot \frac{AB}{DE},$$

$$= \frac{BC}{EF} \cdot \frac{BC}{EF}$$

$$= \frac{BC^2}{EF^2} = \frac{CA^2}{FD^2} = \frac{AB^2}{DE^2}.$$

Exercises.

1. Make a triangle similar to another triangle, and four times its size.

2. Show that two similar triangles are to one another as the squares on the medians from corresponding angles.

3. ABC is a triangle, and AD, CF are perpendiculars from A and C on BC and AB respectively. Show that the triangles ABD, CBF are in the ratio $AB^2 : BC^2$.

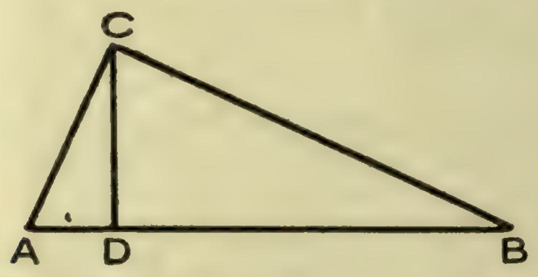
4. ABC is a triangle, and AD, BE, CF are the perpendiculars from the angular points on the opposite sides. Show that the areas of the triangles ABC, DBF are to one another as the squares on AC and DF.

5. If one of two similar triangles has its sides 25 per cent. larger than the corresponding sides of the other, what is the ratio of the areas? What if 25 per cent. less?

6. ABC, DEF are two similar triangles. Find two lines whose lengths are in the same ratio as the areas of these triangles.

PROPOSITION XII. THEOREM.

In a right-angled triangle, if a perpendicular be drawn from the right angle to the hypotenuse, the triangles on each side of it are similar to the whole triangle and to one another.



Let **ABC** be a $\triangle$, having **C** a rt. $\angle$; and let **CD** be $\perp$r to **AB**.

Then the $\triangle$s **ADC, CDB** *are similar to the whole* $\triangle$**ACB**, *and to one another.*

In $\triangle$s **ADC, ACB,**

$\angle$**A** is common,

rt. $\angle$**ADC** = rt. $\angle$**ACB**;

∴ the $\triangle$'s ADC & ACB are equian...

∴ the $\triangle$s **ADC, ACB** are similar.

In like manner the $\triangle$s **CDB, ACB** may be shown to be similar.

Hence the $\triangle$s **ADC, CDB** are similar.

COROLLARY 1. Since the Δs **ADC**, **ACB** are similar,

$$\frac{AD}{AC}=\frac{AC}{AB}, \text{ or } AC^2=AD \cdot AB.$$

In like manner $BC^2=BD \cdot AB$.

COROLLARY 2. Since the Δs **ADC**, **CDB** are similar,

$$\frac{AD}{DC}=\frac{DC}{DB}.$$

Hence **DC** is a *mean proportional* between **AD** and **DB.**

Accordingly, to find a mean proportional between two given lines **AD**, **DB**, place them in the same straight line. On the line which together they form, describe a semicircle, the angle in which may readily be shown to be a right angle. Through **D** draw **DC**, perpendicular to **AB**, to meet the semicircle. **DC** is a mean proportional between **AD** and **DB.**

COROLLARY 3. Evidently $DC^2=AD \cdot DB$, so that the construction in Cor. 2 gives us the means of finding the side of a square equal in area to a given rectangle.

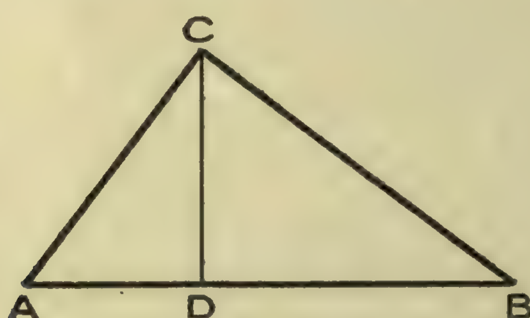
Hence we can describe a square equal to a given triangle, first describing a rectangle equal to the triangle. (II., 11.)

Hence also we can describe a square equal to a given rectilineal figure. (II., 12.)

Relation between Squares on Sides of a Triangle.

PROPOSITION XIII. THEOREM.

In a right-angled triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides.



Let **ABC** be a Δ having the rt. $\angle$ **ACB.**

Then the square on **AB** *is equal to the sum of the squares on* **AC** *and* **CB.**

From **C** draw **CD** perpendicular to **AB.**

Since the Δs **ADC, ACB** are similar, (III., 12.)

$$\frac{AC^2}{AB^2}=\frac{\Delta ADC}{\Delta ACB}.$$

Since the Δs **CDB, ACB** are similar,

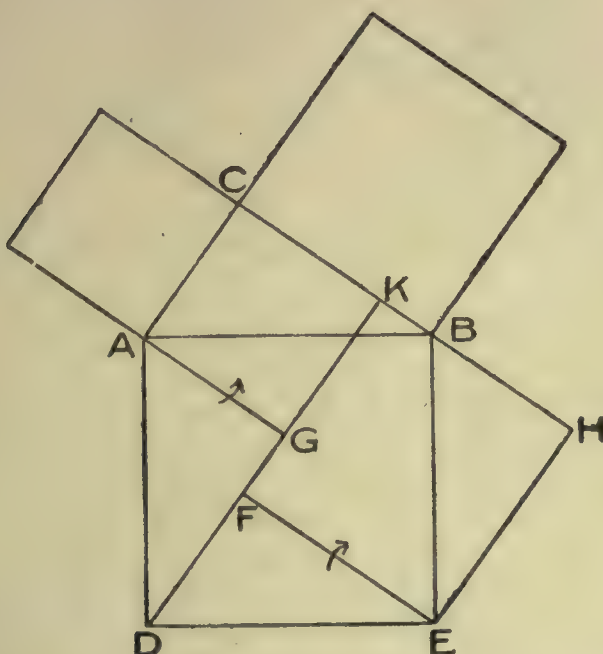
$$\frac{BC^2}{AB^2}=\frac{\Delta CDB}{\Delta ACB}.$$

Hence $$\frac{AC^2+BC^2}{AB^2}=\frac{\Delta ADC+\Delta CDB}{\Delta ACB}.$$

But $\Delta ADC+\Delta CDB=\Delta ACB$;

$$\therefore\ AC^2+BC^2=AB^2.$$

NOTE. The largest square may be cut up so as to form the two other squares, as indicated in the following figure:



The lines **DK, EH** are drawn ∥ to **AC**, and the lines **EF, AG, BH** ∥ to **BC**.

The triangles **AGD, DFE, BHE** are readily shown to be equal in all respects to the triangle **ACB**, and in the process **AK, KE** are shown to be the squares on **AC, CB**, respectively.

The triangle **AGD** may be turned about **A**, in the direction indicated by the arrow-head, into the position **ACB**; the triangle **EFD** may be turned about **E**, in the direction indicated by the arrow-head, into the position **EHB**.

Thus the largest square is transformed into the squares **AK, KE**, which are the squares on **AC, CB** respectively.

Exercises.

1. The side of a right-angled isosceles triangle is to the hypotenuse as 1 to $\sqrt{2}$.

Show how to bisect a triangle by a line parallel to a side.

2. Find a line which bears to a given line the ratio $\sqrt{3}$ to 1, and also the ratio 1 to $\sqrt{3}$.

Given a square, construct a square three times, and also another one-third the given one.

3. Construct a square equal to the sum of three given squares.

4. Describe a square equal to the difference of two given squares.

5. AD is the perpendicular from A on the side BC of the triangle ABC. Show that the difference between the squares on AB, AC is equal to the difference between the squares on BD, DC.

6. ABC is a triangle having the right angle C, and CD is perpendicular to AB. Show that the square on AB exceeds the squares on AD, DB by twice the square on CD.

7. If a quadrilateral has its diagonals at right angles to each other, show that the right-angled triangle formed with one pair of opposite sides has its hypotenuse equal to the hypotenuse of the right-angled triangle formed with the other pair of opposite sides.

8. Divide a given straight line into two segments, such that the square on one segment is double the square on the other.

(Find two lines which are as $\sqrt{2}$ to 1, place them in the same straight line, and divide the given line similarly to this.)

9. ABC is an equilateral triangle, and AD is the perpendicular to BC. Show that the sides of the triangle ABD are in the ratios $1 : 2 : \sqrt{3}$.

10. The sum of the squares on the sides of a rhombus is equal to the sum of the squares on its diagonals.

11. If similar triangles be described on the three sides of a right-angled triangle, the triangle on the hypotenuse is equal in area to the sum of the two other triangles.

12. Divide a straight line into two parts, such that the sum of their squares may be equal to a given square, when possible.

(At end of line make an angle of 45°, etc.)

13. The square on the side opposite the obtuse angle of an obtuse-angled triangle, is greater than the sum of the squares on the sides containing the obtuse angle.

14. If the square on one side of a triangle be less than the sum of the squares on the two other sides, the angle contained by these sides is an acute angle; if greater, an obtuse angle. (Use Prop. ~~18~~, 17 Bk. I.)

15. Given $(2n+1)^2+(2n^2+2n)^2=(2n^2+2n+1)^2$,

by assigning to n in succession the values 1, 2, 3, . . . , form a series of whole numbers, in groups of three, such that each group gives the lengths of the sides of a right-angled triangle.

III, 13

✓ 16. Prove by an application of the Proposition, that if in two right-angled triangles the hypotenuses are equal, and a side in one equal to a side in the other, the triangles are equal in all respects.

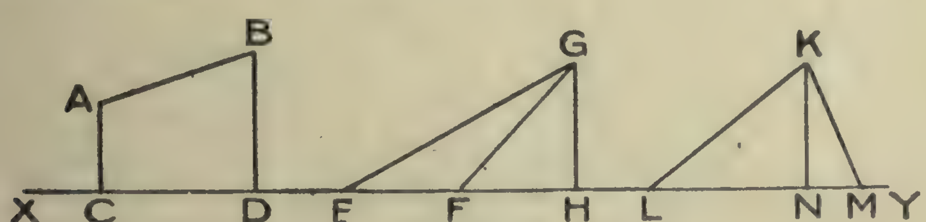
✓ 17. If ABCD be a quadrilateral and $AB^2+CD^2=AD^2+BC^2$, then AC, BD are at right angles.

18. ABCD is a rectangle, and P any point whatever. Show that $PA^2+PC^2=PB^2+PD^2$.

19. If a square and a rectangle have the same area, the perimeter of the rectangle is greater than that of the square.

20. In the triangle ABC a perpendicular is drawn from A to BC, meeting it at D, between B and C. Show that if AD is a mean proportional between BD and DC, the angle BAC is a right angle.

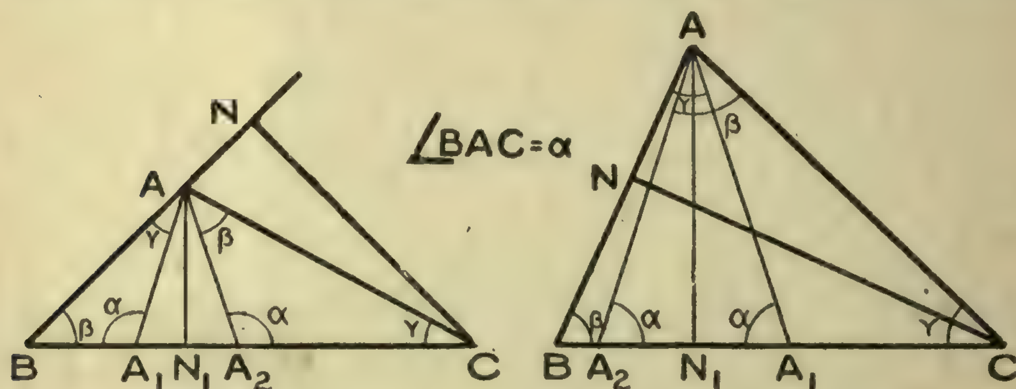
The **projection** of a terminated straight line on any other straight line is the length intercepted between the feet of the perpendiculars from the ends of the terminated line on the other.



Thus the projection on **XY** of **AB** is **CD**, of **EG** is **EH**, of **FG** is **FH**, of **KM** is **NM**, etc. The projections of both **EG** and **FG** on **HG** is **HG** itself; and similarly the projections of both **KL** and **KM** on **KN** is **KN**.

PROPOSITION XIV. THEOREM.

In any triangle the square on the side opposite any angle, according as the angle is obtuse or acute, is greater or less than the sum of the squares on the sides containing the angle, by twice the rectangle contained by either of these sides and the projection upon it of the other.



Let **ABC** be a $\triangle$, and from **C** let **CN** be drawn $\perp$r to **AB**, so that **AN** is the projection of **AC** on **AB**.

Then $BC^2 \mp 2BA \cdot AN = BA^2 + AC^2$, *according as the* $\angle$ **A** *is obtuse or acute.*

At **A** make $\angle BAA_1 = \angle C$, and $\angle CAA_2 = \angle B$, so that the $\triangle$s BA_1A, AA_2C are similar to the $\triangle BAC$, the $\angle$s at A_1, A_2 being $= \angle BAC$.

Thus AA_1A_2 is an isosceles $\triangle$, and the $\perp$r AN_1 bisects it.

$$\text{Then } \frac{\triangle BA_1A}{\triangle BAC} = \frac{BA^2}{BC^2}, \quad \text{(III., 11.)}$$

$$\text{and } \frac{\triangle AA_2C}{\triangle BAC} = \frac{AC^2}{BC^2};$$

$$\therefore \frac{\triangle BA_1A + \triangle AA_2C}{\triangle BAC} = \frac{BA^2 + AC^2}{BC^2};$$

$$\text{or } \frac{\triangle BAC \mp \triangle AA_1A_2}{\triangle BAC} = \frac{BA^2 + AC^2}{BC^2};$$

$$\text{or } 1 \mp \frac{2A_1N_1}{BC} = \frac{BA^2 + AC^2}{BC^2}; \quad \text{(III., 1.)}$$

$$\text{or } BC^2 \mp 2A_1N_1 . BC = BA^2 + AC^2.$$

But $\frac{A_1N_1}{AN} = \frac{AN_1}{CN}$, from similar $\triangle$s AA_1N_1, CAN;

$$= \frac{BA}{BC}, \ \triangle\text{s } CNB,\ AN_1B \text{ being similar.}$$

$$\text{Hence } A_1N_1 . BC = BA . AN,$$

$$\text{and } BC^2 \mp 2BA . AN = BA^2 + AC^2.$$

NOTE. The preceding demonstration includes Prop. XIII., for if the $\angle BAC$ be a right angle, the triangle AA_1A_2 vanishes, and we have $1 = \frac{BA^2 + AC^2}{BC^2}$. Indeed the above proof shows that 47th, Book I., and 12th and 13th, Book II., of Euclid, may all be regarded as one and the same proposition. It has been thought well, however, from its extreme simplicity, to make a separate proposition of the case in which the angle referred to in the enunciation is a right angle.

Proposition 12 of "Additional Propositions" after Book V., is an alternative proof of this theorem, and may be substituted for the preceding demonstration.

The construction in the Proposition suggests generalizations of certain propositions which are usually stated with reference to right-angled triangles. Thus AB is a mean proportional between CB and BA_1; AC is a mean proportional between BC and CA_2; AA_1 $(= AA_2)$ is a mean proportional between BA_1, and A_2C.

The proof is shortened if the truth stated in Ex. 1, page 128, be assumed; for then, after

$$BC^2 \mp 2A_1N_1 . BC = BA^2 + AC^2,$$

from similar $\triangle$s BA_1A, BAC,

we have $\dfrac{A_1N_1}{AN} = \dfrac{BA}{BC}$, or $A_1N_1 . BC = BA . AN$,

and $BC^2 \mp 2BA . AN = BA^2 + AC^2$.

COROLLARY. **If the sum of the squares on two sides of a triangle be equal to the square on the third side, the angle opposite this third side is a right angle,** for when the angle is obtuse or acute, the Proposition shows that another equation holds.

Exercises.

1. In a triangle in which A is an acute angle, and a, b, c are the lengths of the sides opposite to A, B, C respectively, show that

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

2. If ABC be a triangle, and AD be the median to the middle point of BC, show that

$$AB^2 + AC^2 = 2AD^2 + 2DC^2.$$

3. If AD, BE, CF be the three medians of a triangle ABC, show that

$$3(AB^2 + BC^2 + CA^2) = 4(AD^2 + BE^2 + CF^2).$$

4. The sum of the squares on the sides of a parallelogram is equal to the sum of the squares on the diagonals.

5. In a quadrilateral the sum of the squares on the diagonals is twice the sum of the squares on the straight lines joining the middle points of opposite sides.

6. In the figure of Prop. xiv., if the angle BAC becomes more and more obtuse, until A finally coincides with BC, what does the proposition become?

7. The sides of a triangle are 5, $2\sqrt{13}$, 9. Find the perpendicular on the longest side from the opposite angle, and thence the area of the triangle.

8. If a, b, c be the sides of a triangle, and p be the perpendicular on c, show that

$$p^2 = b^2 - \left(\frac{b^2+c^2-a^2}{2c}\right)^2 = \frac{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}{4c^2}.$$

Hence show that

$$4 \text{ area} = \sqrt{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}.$$

9. The sum of the squares on the sides of any quadrilateral is equal to the sum of the squares on the diagonals, together with four times the square on the straight line joining the middle points of the diagonals.

10. Construct a triangle, having given its base, its area, and the sum of the squares on its sides. (By Ex. 2, the median to centre of base is given.)

11. If from an end of the base of an isosceles triangle a perpendicular be drawn to the opposite side, then twice the rectangle contained by that side and its segment adjacent to the base, is equal to the square on the base.

12. Find the locus of a point which moves so that the sum of the squares on lines joining it to two given points is constant. (Use Ex. 2.)

13. What is the magnitude of the obtuse angle of a triangle, when the square on the side opposite the obtuse angle is greater than the sum of the squares on the sides containing it, by the rectangle contained by these two sides?

14. If squares be described on the three sides of any triangle, and perpendiculars from the angles on the opposite sides be continued so as to divide each square into two rectangles, then any two rectangles having angular points at the same angle of the triangle are equal.

---

15. Make the figure and go through the demonstration of Prop. xiv., when the angle at A is less than each of the angles at B and C.

16. If a, b be the sides of a right-angled triangle which contain the right angle, and p the perpendicular from the right angle on the hypotenuse, then $\dfrac{1}{a^2}+\dfrac{1}{b^2}=\dfrac{1}{p^2}$.

17. If the medians of a triangle ABC intersect in O, prove that

$$AB^2+BC^2+CA^2=3(OA^2+OB^2+OC^2).$$

18. Through a given point O draw three lines OA, OB, OC of given lengths, such that the points A, B, C may be in the same straight line, and one of these points equidistant from the other two.

19. ABC is a triangle, and FE, varying in position, is drawn parallel to the base BC. Show that the locus of the intersection of BE, CF is the median through A bisecting BC.

20. If the bisectors of the angles A, C, of a quadrilateral ABCD meet on the diagonal BD, show that the bisectors of the angles B, D, meet on the diagonal AC.

21. ABC is a triangle, and the perpendiculars AD, BE, CF, from A, B, C on the opposite sides, meet in O. Show that $AO.OD = BO.OE = CO.OF$.

22. D, E, F are points on the sides of the triangle ABC. The angles which FD, ED make with BC are equal; likewise those which DE, FE make with CA; and those which EF, DF make with AB. Show that AD, BE, CF are perpendicular to BC, CA, AB, respectively. (Prove that equal angles at D are each $= \angle A$, etc. Then show that angles BFC, BDA are equal, etc.)

23. ABCD is a quadrilateral whose diagonals AC, BD, intersect in O, and the angles ABD, ACD are equal. Prove that the triangles OAD, OBC are equiangular.

24. A triangle ABC, whose angles are of given magnitude, has A at a fixed point, and C moving along a fixed line EF. Show that the locus of B is a fixed line. (Construct a triangle ABC with angles of given magnitude, and having BC coincident with EF. Then the locus required is the line through B, making an angle $B+C$ with EF.)

25. If two triangles are to one another as the squares on their bases, and have an angle in one adjacent to the base equal to an angle in the other adjacent to the base, the triangles are equiangular.

26. If two isosceles triangles are to one another as the squares on their bases, show that the triangles are equiangular.

27. A, B, C are three fixed points. Through C draw a straight line so that the parts of it intercepted between C and the perpendiculars on the line from A and B, may be in a given ratio.

28. ABC is a triangle, and the angle BAC is bisected by AD, which meets BC in D. The middle point of BC is E. Show that ED is to EB as the difference between Δs ADC, ADB to Δ ABC.

BOOK IV.

THE CIRCLE.

INTRODUCTION: SYMMETRY.

CHORDS AND RADIUS-VECTORS.

ANGLES IN A CIRCLE.

TANGENTS.

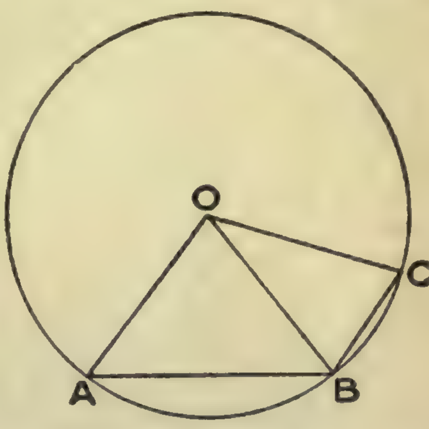
SEGMENTS OF INTERSECTING CHORDS.

Introduction: Symmetry.

1. The circle has already been defined, as well as its various parts and some of the lines that are associated with it.

Having the notion of locus, we may define the circle as the locus of a point which moves so as to be at a constant distance from a fixed point.

2. In considering the form of the curve, we readily see that **a straight line cannot meet it in more than two points,** or, **no three points on it can be in the same straight line;** for example, the three points A, B and C. For each angle at B, being an angle of an isosceles triangle, is less than a right angle, and therefore their sum is less than two right angles, and BC is inclined to AB at an angle less than 180°. This result is true, however small AB and BC may be, which shows why we say **the circle is everywhere concave to its centre.** Of course we have here proved that **no part of the circumference of a circle can be a straight line.**



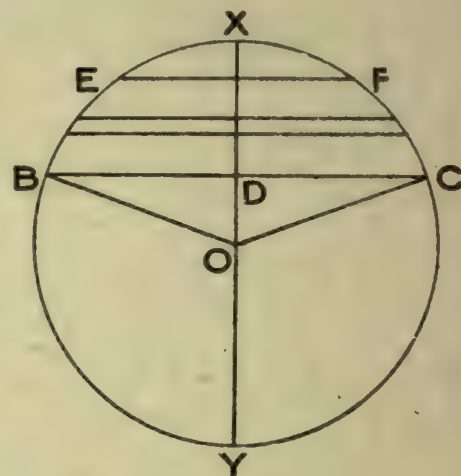
3. The fundamental quality of the circle, next to the equality of its radii, and as a consequence of the equality of its radii, is its **symmetry.**

Central Symmetry.

4. In the first place, every line drawn through the centre, from circumference to circumference, *i.e.*, **every diameter, is bisected at the centre.** This is called **central symmetry.**

Other figures than the circle have this central symmetry. Thus every line through the intersection of the diagonals of any parallelogram, and terminated by the periphery, is bisected at such intersection.

Axial Symmetry.



5. In the second place, **every chord drawn at right angles to a diameter is bisected by that diameter.** This is called **axial symmetry,** *i.e.*, symmetry with respect to an axis. Thus let the chord **BDC** be perpendicular to the diameter **XY**; then **BD** = **DC**.

For, in Δs **ODB**, **ODC**,

rt. $\angle$s at **D** are equal;

also $\because$ **OB** = **OC**, $\angle$ **OBD** = $\angle$ **OCD**;

$\therefore$ $\angle$ **DOB** = $\angle$ **DOC**.

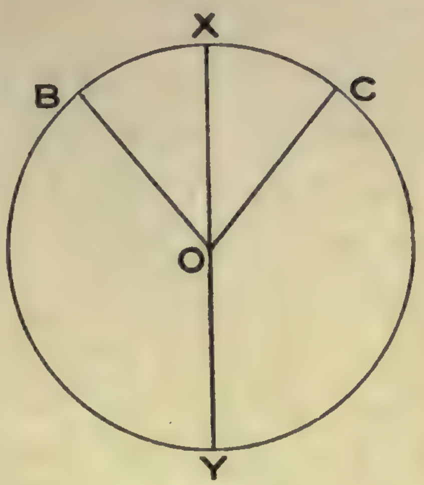
Also sides **DO**, **OB** = sides **DO**, **OC**;

$\therefore$ **BD** = **DC**. (I., 6.)

Thus wherever there is on the circumference a point **B** on one side of a diameter, directly opposite, on the other side of the diameter, and at the same distance from it, is another point **C**, also on the circumference.

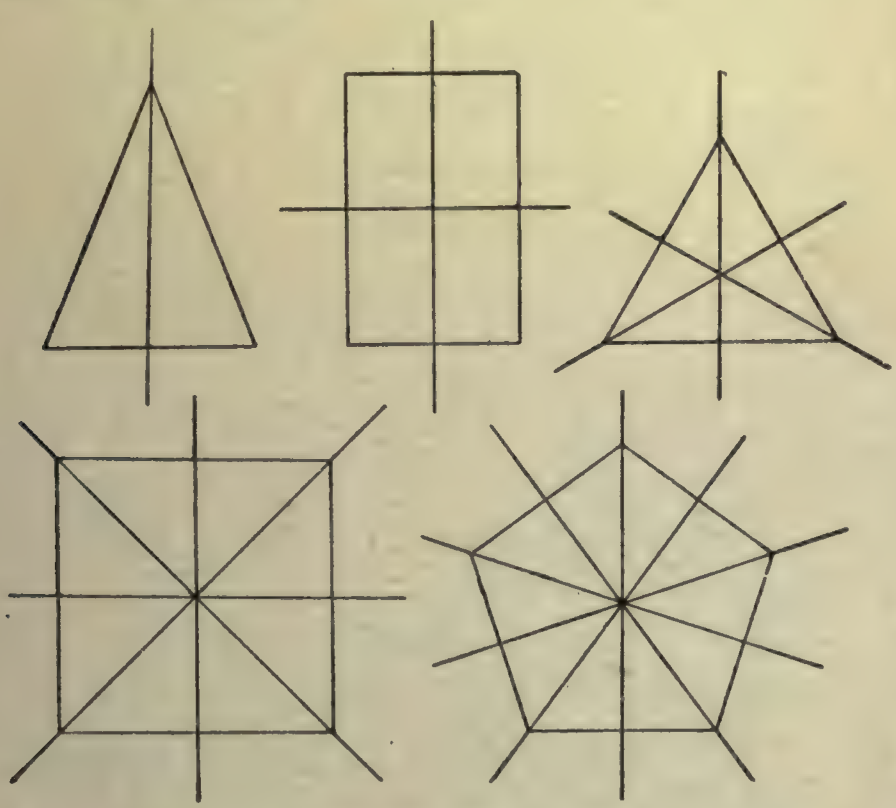
Alternative Proof of Preceding Theorem:

Suppose the circle folded about the diameter **YX**. Then the point **B** must fall on the circumference at the other side of **XY**; for if it fell outside or inside, we should have radii of unequal lengths. Let **B** fall on **C**. When the circle is unfolded, **BC** must be at right



angles to **XY**, for if it cut **XY** obliquely, the process of folding would not bring **B** and **C** together. Also, that **B** and **C** may fall together, **BC** must be bisected by **XY**. That is, **XY** bisects **BC** at right angles.

6. This symmetry with respect to a diameter, holds with respect to every diameter. Thus there is an infinite number of axes with respect to which the circle is symmetrical and the circle may be said to have infinite axial symmetry, or to be infinitely symmetrical. *It is in this infinite axial symmetry that the circle differs from all other figures.*

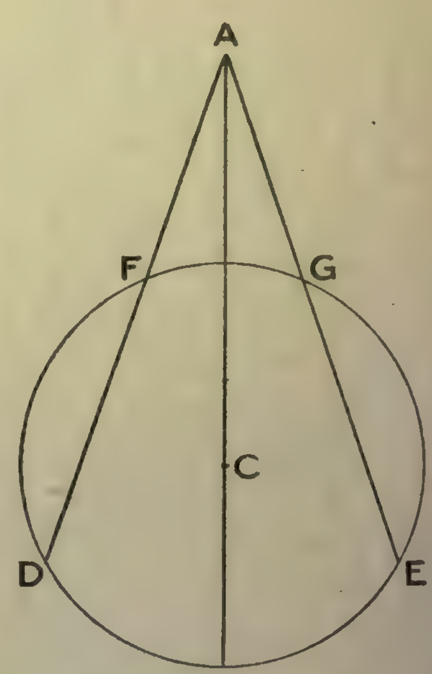


Other figures enjoy axial symmetry to a limited degree, *i.e.*, with respect to a limited number of axes. Thus the isosceles triangle is symmetrical with respect to one axis; the rectangle with respect to two; the equilateral triangle with respect to three; the square with respect to four; the regular pentagon with respect to five; and so on. The circle alone is symmetrical with respect to an infinite number.

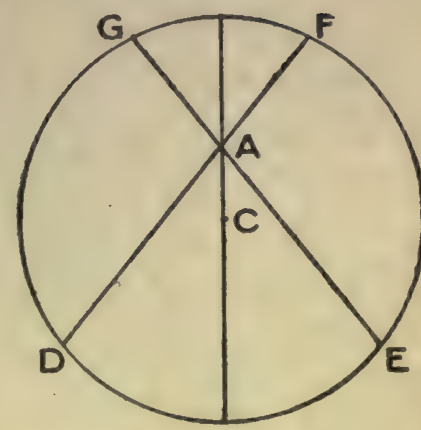
The following consequences of this axial symmetry may be noted:

7. If a circle be folded about any diameter, one semi-circumference completely coincides with the other, each point on the circumference coinciding with the corresponding point on the opposite side of the diameter, for each chord perpendicular to the diameter folds over on itself, its ends coinciding. In axial symmetry, the point which corresponds to another is often spoken of as its *image.*

8. If from a point without a circle a line be drawn to the centre, and on opposite sides of this, making equal angles with it, lines be drawn to the circumference, the corresponding segments of these lines are equal. Thus if the angles at **A** be equal, **AD** and **AE** are equal, and also **AF**, **AG**. For, folding the figure about **AC**, **AD** falls on **AE**, because the angles at **A** are equal, and one semi-circumference on the other. Hence the points **F** and **G** coincide, and also **D** and **E**, since the figure on one side of **AC** coincides completely with the figure on the other side.



In like manner, if the point **A** be within the circle, and the angles at **A** be equal, on folding the circle about

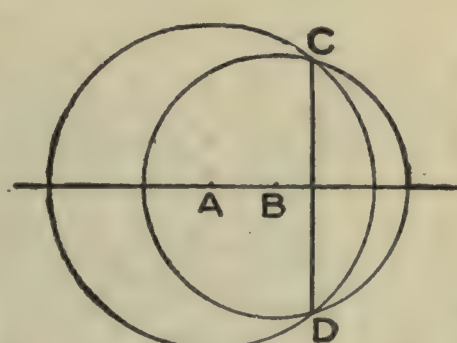


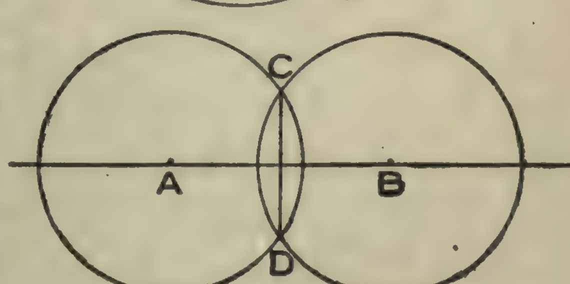
AC, F and G must coincide, and also E and D. Thus AF = AG and AE = AD.

9. The converse to the proposition that a diameter bisects all chords at right angles to it, is also true, viz.,—**a straight line bisecting any chord at right angles is a diameter,** for such a line is the locus of all points at equal distances from the ends of the chord, and therefore must pass through the centre. It follows that **a diameter through the bisection of a chord is at right angles to it.**

10. **If two circles intersect they cannot have the same centre,** for then having equal radii (the line from common centre to point of intersection) and a common centre, they would be entirely coincident.

Also if two circles intersect in one point they must intersect in another, since they are closed figures.



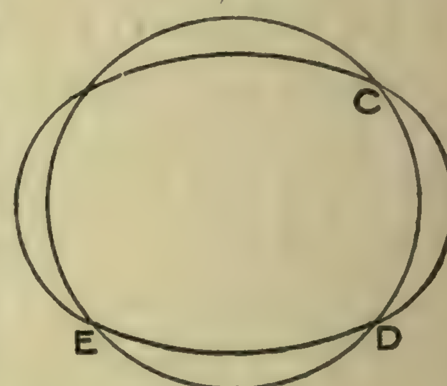


11. Let the two circles whose centres are A and B, intersect. The line through A and B, since on it lie diameters of both circles, is an axis of symmetry for both circles. Hence if C be a point of intersection, *i.e.*, a point common to both circles, then C's "image" on the other side of AB must also be a point common

to both circles. Hence,—**If two circles intersect, the line joining their centres bisects at right angles the line joining their points of intersection.**

Of course, conversely, as has already been shown, **the line bisecting at right angles the common chord passes through both centres** (§ 9).

12. It follows that **two circles cannot ~~have~~ more than two points ~~common~~,** for if they had three common points, **C, D** and **E,** then *both* centres would lie in the lines bisecting **CD** and **DE** at right angles, and as these lines can have only one common point, we should have intersecting circles with a common centre, which (§ 10) has been shown to be impossible.

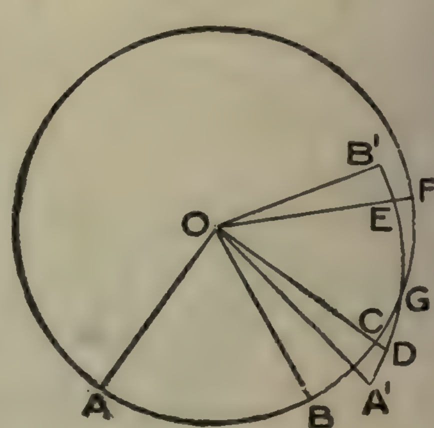


Homogeneity of Circumference.

13. Thirdly, there is an important correspondence between an arc of a circle and the rest of the circumference of the same circle, which may be stated as follows:

If an arc of a circle be supposed rigidly connected with the centre, and to rotate about the centre, it continues, during such motion, to coincide throughout its entire length with the circumference.

Thus, if the arc **AB** be supposed rigidly connected with the centre **O,** and to rotate about **O,** it will, throughout its entire length, continue to coincide with the circumference.

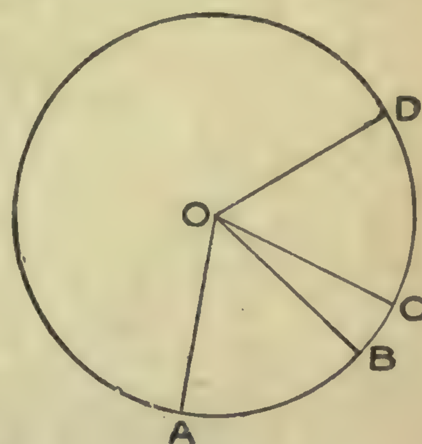


For no part of the arc, in any new position, could fall outside the circumference, as **A′G,** since then we should have radii **OC, OD** of the same circle unequal. Nor can any part of the arc fall within the circumference for the same reason. Nor can the arc cut the circumference, as at **G,** for then part of it would lie without the circumference, and part within.

Hence we may say that the circumference of any (the same) circle is homogeneous,—each part is the same as the rest.

Important consequences follow from this:

14. **In the same circle, equal arcs subtend equal angles at the centre.** For let **AB, CD** be equal arcs, and let **AB** be turned about **O** until **A** coincides with **C**; then **B** will coincide with **D,** and the angles **AOB, COD** coincide with one another and are equal.



Conversely, **in the same circle if two angles at the centre be equal, they are subtended by equal arcs.** For if the angles **AOB, COD** be equal, rotating **AOB** about **O,** when **OA** coincides with **OC, A** coincides with **C, OB** coincides with **OD,** and **B** with **D.** Hence the arcs **AB, CD** coincide, and are equal.

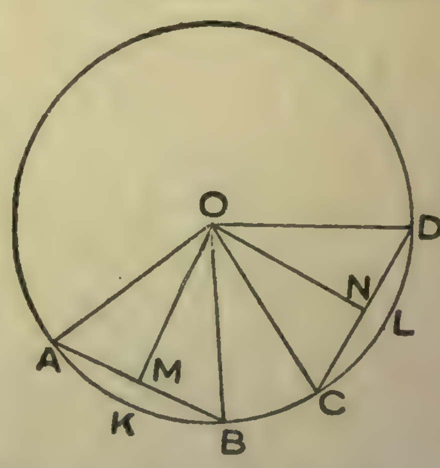
15. **In the same circle, equal chords cut off equal arcs.** For if the chords **AB, CD** be equal, the triangles **OAB, OCD,** having all their other sides

equal, are equal in all respects. Turning, then, **OAB** about **O**, when **A** coincides with **C**, **B** coincides with **D**, and the arcs **AKB**, **CLD** coincide and are equal.

Conversely, **in the same circle, if two arcs be equal their chords are equal.** For if the arcs **AKB**, **CLD** be equal, turning **AKB** about **O**, when **A** coincides with **C**, **B** coincides with **D**, and the chords **AB**, **CD** coincide and are equal.

16. Evidently also, **equal chords in a circle are equally distant from the centre.** For if the chords **AB**, **CD** be equal, the triangles **OAB**, **OCD**, having all their other sides equal, are equal in all respects. Turning, then, the triangle **OAB** about **O**, it can be made to coincide completely with **OCD**, and the perpendiculars from **O** on the coincident bases; coincide and are equal.

Conversely, **chords which are equally distant from the centre are equal to one another.** For, let the chords **AB**, **CD** be at equal distances, **OM**, **ON**, from the centre. Let the figure **OAKB** rotate about **O**, until **OM** coincides with **ON**, and **M** with **N**. Then **AB** and **CD** will fall together, being both perpendicular to the same line; also **A** and **C** must coincide, and **B** and **D**; otherwise we should have the same straight line **CD** (**AB**) meeting the circumference in more than two points.



17. If two circles have equal radii, they are equal; for, placing them so that their centres coincide, their circumferences will also coincide, because the radii are equal.

N.B. The preceding six properties, though demonstrated of the same circle, are also true of equal circles, as is evident if the circles be placed in coincidence.

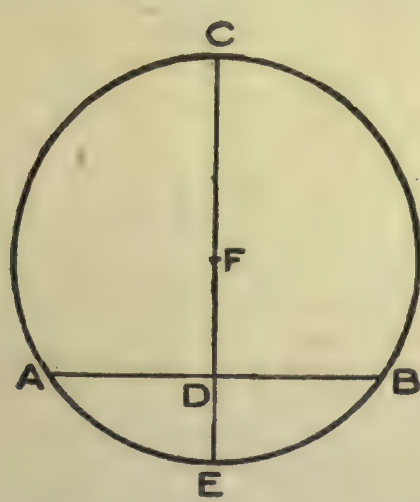
Chords and Radius-Vectors.

A straight line drawn from a point, called the *origin* or *pole*, to a line, straight or curved, is called a *radius-vector*.

Thus **AD, AE, AF, AG** (§ 8, pp. 154–5) are radius-vectors, **A** being the origin or pole.

PROPOSITION I. PROBLEM.

To find the centre of a given circle.



Let **ABC** be the given circle.

It is required to find its centre.

Draw any chord **AB**; bisect it at **D**; from **D** draw **EDC** at rt. $\angle$s to **AB**, meeting the circumference at **C** and **E**; bisect **CE** in **F**. Then **F** is the centre of the circle.

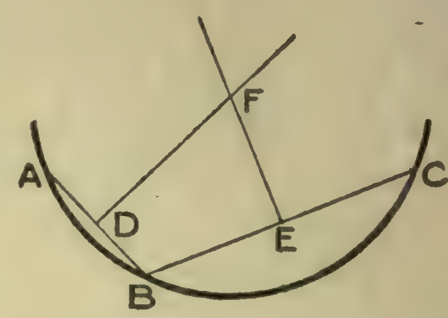
Because **EC** bisects **AB** at rt. $\angle$s; $\therefore$ **EC** is the locus of all points equally distant from **A** and **B**.

$\therefore$ the centre lies in **EC**;

and $\therefore$ must be at **F**, its point of bisection.

COROLLARY. If only an arc of the circle be given, so that the line which bisects the chord at rt. $\angle$s may not meet the arc in two points, the following construction will give the centre:

Draw two chords **AB, BC**; bisect them at right angles by the lines **DF, EF**, intersecting in **F**. Then the centre lies on each of the lines **DF, EF**; and therefore must be at **F**, where they intersect.



This gives the solution of the problem,—An arc, or a segment of a circle being given, to describe the circle of which it is the arc, or segment; for, the centre being found, the circle may be completed.

Exercises.

NOTE. In the exercises, the results of the preliminary discussion on the symmetry of the circle may of course be employed.

1. The locus of the centres of all circles which pass through two given points, is a straight line.

2. The locus of the middle points of all parallel chords of a circle, is the diameter perpendicular to the chords.

3. The straight lines bisecting at right angles the sides of a quadrilateral inscribed in a circle, all pass through the same point.

4. Describe a circle with a given point as centre, and cutting a given circle at the ends of a diameter.

5. AB, AC are two equal chords in a circle; show that the straight line which bisects the angle BAC passes through the centre.

6. Describe a circle which shall pass through three given points that are not in the same straight line.

7. Describe, when possible, a circle that shall pass through two given points, and have a given radius.

8. Through a given point within a circle, draw a chord that shall be bisected at that point.

9. The parts of a straight line intercepted between the circumferences of two concentric circles are equal.

10. The line joining the middle points of two parallel chords in a circle, passes through the centre.

11. Describe a circle which shall pass through two given points and have its centre in a given line.

12. Two chords of a circle, AB, AC, make equal angles with the radius through A. Show that AB = AC.

13. If two circles cut one another, any two parallel straight lines drawn through the points of intersection, and terminated by the circles, are equal.

14. The greater of two chords in a circle subtends the greater angle at the centre.

15. Find the shortest distance between the circumferences of two circles which do not meet.

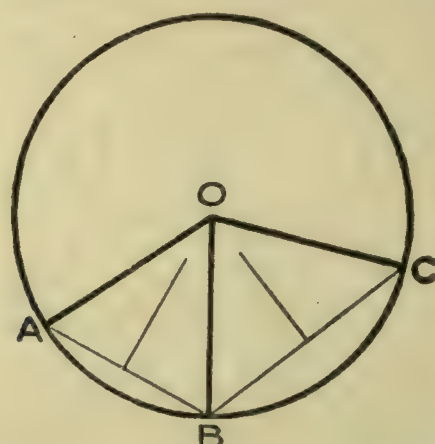
16. Find the shortest distance between the circumferences of two concentric circles.

17. ABCD is a parallelogram, and a circle is described to pass through the three points A, B, C (Ex. 6). Show that it cannot pass also through D unless the parallelogram be a rectangle.

18. If from a point without a circle, two equal lines be drawn to the circle and produced, they are equally distant from the centre.

PROPOSITION II. THEOREM.

If from a point within a circle more than two equal straight lines can be drawn to the circumference, that point is the centre of the circle.



Let **ABC** be the circle, and let the three lines **OA, OB, OC,** from the point **O** within the circle, to the circumference, be equal.

Then **O** *must be the centre of the circle.*

Join **AB, BC.**

Because **OAB, OBC** are isosceles △s, the st. lines bisecting at rt. ∠s the bases **AB, BC,** must pass through the vertex of each, which is the point **O.** (Loci, Ex. 2.)

But the st. lines bisecting at rt. ∠s the chords **AB, BC** intersect in the centre of the circle (IV., 1, Cor.);

∴ **O** is the centre.

Exercises.

1. Two circles cannot meet in three points without coinciding entirely.

2. Through three points, which are not in the same straight line, only one circle can be drawn.

3. Two circles cannot have a common arc without coinciding entirely.

---

4. If equal chords of two circles subtend equal angles at the centres, the two circles must be equal.

5. Two triangles are inscribed in a circle, and two sides of one are equal to two sides of the other, centre being in both cases in or without $\angle$s between these sides. Prove third sides equal.

6. Show how to bisect any given arc of a circle.

7. Divide the circumference of a circle into twelve equal arcs.

8. The middle points of all chords of a circle, which are of the same length, lie on a concentric circle.

9. Given two chords of a circle, in magnitude and position, describe the circle.

10. Given a chord of a circle, in magnitude and position, and a line in which the centre lies, describe the circle.

11. Given the centres and positions, but not the magnitudes, of two chords of a circle, also a point through which the circle passes, describe the circle.

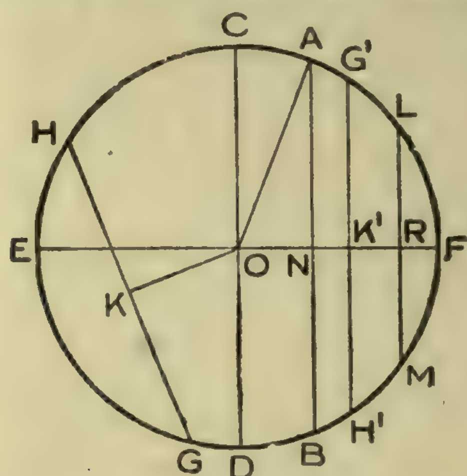
12. If a parallelogram be inscribed in a circle, show that each diagonal is a diameter of the circle.

13. The centre of a chord of a circle being given, and the radius of the circle, determine the area within which the centre may lie.

14. In a given circle draw a chord of given length, not greater than the diameter, so that its centre may be on a given chord.

PROPOSITION III. THEOREM.

If a chord of a circle move parallel to itself from the centre to the extremity of the diameter to which it is always at right angles, it continually decreases and ultimately vanishes.



Let the chord **AB**, of the circle **ABC**, move parallel to itself from the centre **O** to the extremity of the diameter **EF**, to which it is always at right angles.

Then **AB** *continually decreases and ultimately vanishes.*

Join **OA**.

Then since $\angle$ **ONA** is a rt. $\angle$

$\therefore ON^2 + NA^2 = OA^2$.

But OA^2 is always of the same magnitude,

since **OA** is radius of the circle.

Hence $ON^2 + NA^2$ is always of the same magnitude.

But ON^2 continually increases since **ON** does;

$\therefore NA^2$ continually decreases;

and $\therefore$ **NA** continually decreases.

Hence the whole chord **AB** continually decreases.

Again, since $ON^2 + NA^2 = OA^2$,

when **ON** becomes **OF**, $= OA$, then $ON^2 = OA^2$;

and NA^2 must vanish;

and $\therefore$ also **NA**, and **AB**, must vanish.

COROLLARY 1. Since the chord decreases as it moves from the centre in any direction, it is greatest when it passes through the centre, or **the diameter is the greatest chord in a circle.**

COROLLARY 2. If **GH** be a chord nearer to the centre than **LM**, *i.e.*, the perpendicular **OK** less than **OR**; then let **GH** with **OK** rotate until **OK** falls on **OF**, and **GKH** takes the position **G′K′H′**. **G′K′H′** and **LRM** are parallel, and ∴ by the Proposition **G′H′** is greater than **LM**, since it is nearer to the centre. Hence **a chord which is nearer to the centre is greater than one more remote.**

COROLLARY 3. Conversely, if **GH** be given greater than **LM**, on rotating it into the position **G′K′H′**, it must be nearer to the centre than **LRM**, since, by the Proposition, the chord decreases as it recedes parallel to itself from the centre. Hence **the greater chord is nearer to the centre than the less.**

Exercises.

1. Through a given point within a circle draw the shortest chord.

2. In a given circle draw a chord of given length, not greater than the diameter, so that one third may be cut off by a given chord.

3. What is the locus of the middle points of equal chords of a circle?

4. If the shortest chords which can be drawn through the points A, B within a circle, are equal, show that A, B are equally distant from the centre.

5. Through a given point, within or without a given circle, draw a chord of length equal to that of a given chord.

(If A be given point, and C centre, take B in chord so that CB = CA; draw CN perpendicular to chord, and construct triangle on CA equal to triangle CBN.)

6. Through a point A within a circle, two chords are drawn equally inclined to the diameter through A. Show that these chords are equal.

7. Two equal chords, AB, CD, of a circle intersect in O. Show that AO is equal to CO or OD.

8. Through two points on a diameter of a circle equally distant from the centre, two parallel chords are drawn; show that the chords are equal, and that they are divided by the diameter into equal segments.

9. If, in the figure of the preceding question, through the points on the diameter, another pair of parallel chords be drawn, a line joining the ends of the chords through one point is parallel to a line joining the ends of the chords through the other point.

10. In a given circle draw a chord equal to one given chord and parallel to another.

11. Through the bisection of the line joining the centres of two equal circles, a straight line is drawn cutting the circles; show that the circles intersect equal chords in it.

12. PQ is a fixed chord in a circle, and AB is any diameter; show that the sum or difference of the perpendiculars let fall from A and B on PQ, for all positions of AB, is equal to twice the distance of the chord from the centre, and therefore constant.

13. Are we ever certain of the magnitude or position of a circle from knowing the magnitudes of any number of its chords?

14. A chord of a circle is given in magnitude, and a point through which the circle passes. Find the locus of the centre of the least circle that can be described with these data.

15. The centre of a circle is given, and the length of the shortest chord that can be drawn through a given point. Describe the circle.

16. If two points be given and the lengths of the shortest chords that can be drawn through these points, the chords being coterminous, show that the centre of the circle occupies one of two positions.

When does the construction become impossible?

PROPOSITION IV. THEOREM.

If one end of a straight line be fixed and the other end move on the circumference of a circle, the line is greatest when it contains the centre, and least when, on being produced, it passes through the centre; and as the moving end of the line advances from the latter to the former position, the line continually increases.

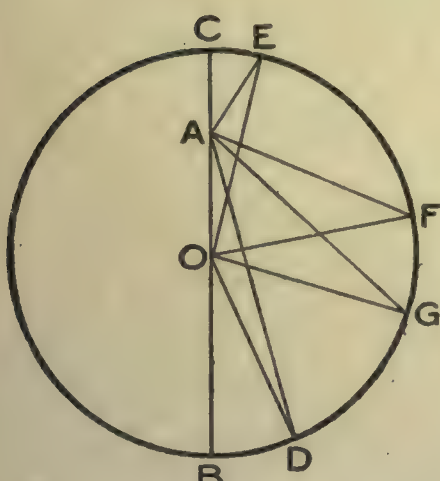


FIG. 1.

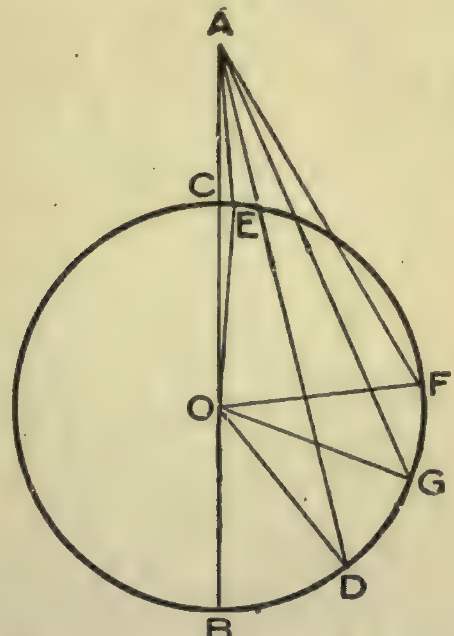


FIG. 2.

Let **A** be the fixed end of the line, and **CFB** the circle on the circumference of which the other end moves.

Then the line is greatest in the position **AB** *when it contains the centre,* **O**, *and least in the position* **AC** *when, on being produced, it passes through the centre; and as the moving end of the line advances along the circumference from the position* **C** *to the position* **B**, *the line continually increases,—thus* **G** *being a point on the circumference farther from* **C** *than* **F** *is,* **AG** *is greater than* **AF**.

Let **AD**, **AE** be any two positions of the line.

Join **OD**, **OE**, **OF**, **OG**.

Then $AB = AO + OB = AO + OD > AD$.

Hence the moving line is greatest in the position **AB**, which contains the centre.

Again,—

Fig. 1.	Fig. 2.
$OC = OE < OA + AE$.	$AO < AE + EO$.
Take away the common part **OA**, and	Take away the equal parts **CO**, **EO**, and
$AC < AE$.	$AC < AE$.

Hence the moving line is least in the position **AC** which, on being produced, passes through the centre.

Again, because in $\triangle$s **AOG**, **AOF**,

AO, **OG** are = **AO**, **OF**, respectively;

but $\angle AOG > \angle AOF$;

$\therefore AG > AF$.

And therefore as the moving end of the line advances along the circumference from **C** to **B**, the line continually increases.

NOTE. The proposition holds good if **A** be on the circumference. **AC** then becomes zero, and **AB** becomes the diameter; and the continual increase of the line becomes the continual increase of the chord of a circle as it gets nearer and nearer to the centre.

Angles in a Circle.

PROPOSITION V. THEOREM.

The angle at the centre of a circle is double of any angle at the circumference, standing on the same arc.

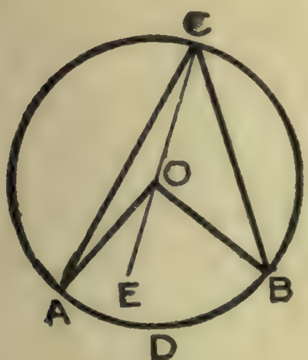


FIG. 1.

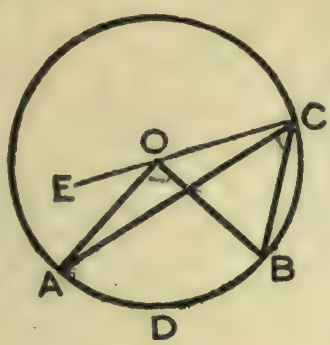


FIG. 2.

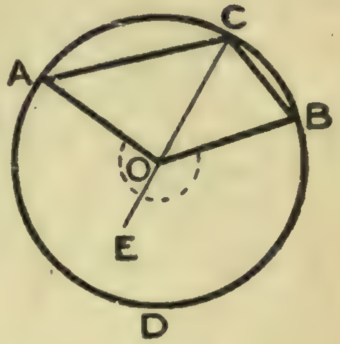


FIG. 3.

Let ABC be a circle with centre O, and let ADB be an arc, and C any point on the rest of the circumference.

Then the $\angle$AOB *at the centre is double the* $\angle$ACB *at the circumference.*

Join CO, and produce it to E.

Then $\because$ OA = OC, the $\angle$OAC = $\angle$OCA. I. 2

But $\angle$AOE = $\angle$OAC + $\angle$OCA;

$\therefore$ $\angle$AOE = twice $\angle$OCA.

Similarly $\angle$BOE = twice $\angle$OCB.

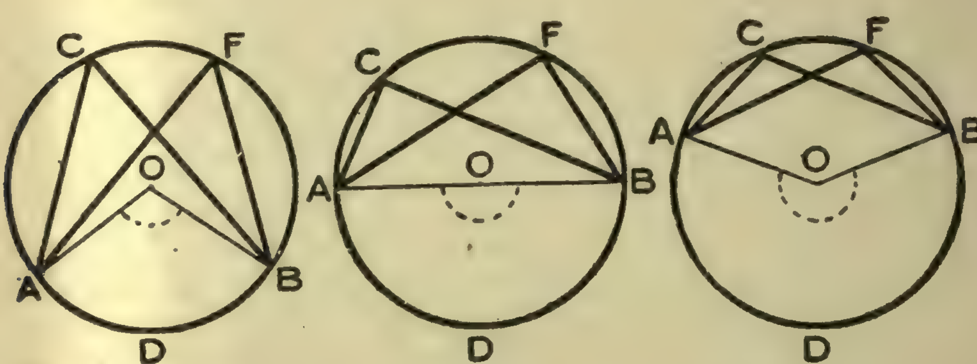
Hence, adding in Figs. 1 and 3, and subtracting in Fig. 2,

$\angle$AOB = twice $\angle$ACB.

NOTE. In Fig. 3 the $\angle$AOB referred to, is the reflex angle at O, which in the figure is marked.

PROPOSITION VI. THEOREM.

Angles in the same segment of a circle are equal.



Let **ABC** be a circle; and let **ACB, AFB** be $\angle$s in the same segment.

Then the $\angle$s **ACB, AFB** *are equal.*

Let **O** be the centre of the circle.

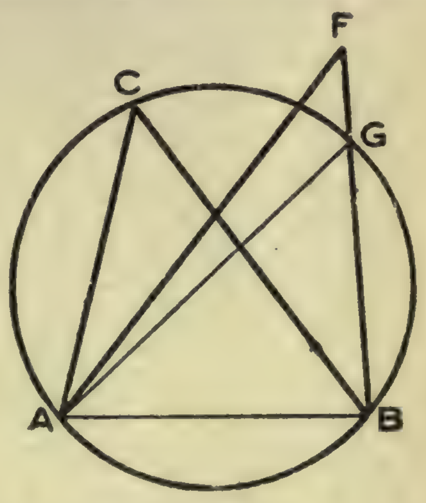
Join **OA, OB.**

Then the $\angle$**AOB** at the centre is double of each of the $\angle$s **ACB, AFB** at the circumference; (IV., 5.)

$\therefore$ the $\angle$s **ACB, AFB** are equal.

The converse of this proposition is true,—

The vertices of all equal angles, standing on the same base and on the same side of it, lie on a segment of a circle passing through the extremities of the base.



For, let **ACB, AFB** be two of the equal $\angle$s on the same base **AB,** and on the same side of it.

Let a circle be described to pass through the three points **A, C, B** (Loci, Ex. 4); and if this does not pass through **F,** let it cut **BF** in **G.**

Join **AG.**

Then $\angle$**ACB** = $\angle$**AFB.** (Hyp.)

Also $\angle$**ACB** = $\angle$**AGB;** (IV., 6.)

$\therefore$ $\angle$**AFB** = $\angle$**AGB,**

which is impossible.

Hence circle through **A, C, B,** must also pass through **F.**

In the same circle (or in equal circles), **equal angles at the circumference stand on equal arcs.** For these arcs subtend equal angles at the centre, since the angles at the circumference are equal; and therefore (§ 14) the arcs are equal.

Also **equal arcs subtend equal angles at the circumference.** For, the arcs being equal, the angles at the centre are equal (§ 14), and therefore the angles at the circumference are equal.

Exercises.

1. What special case of Prop. V. occurs when OA and OB are in the same straight line?

2. Show how to divide a given circle into two segments, so that the angle in one shall be double the angle in the other.

3. C is any point on the arc of a segment whose chord is AB. Show that the sum of the angles CAB, CBA is constant.

4. A circle is divided into two segments by a chord equal to the radius. What is the angle in each segment? What relation do you note between these two angles?

5. ABCD is a quadrilateral in a circle, and the sides AB, CD are equal. Show that BC is parallel to AD, and that the diagonals are equal.

6. Given the base and vertical angle of a triangle, find the locus of the vertex.

7. Two circles intersect at A and B, and a chord XAY, is drawn terminated both ways by the circumferences. Show that the angle XBY is constant.

8. AB, CD are two chords of a circle which intersect at E. Show that the triangles AED, CEB are equiangular, and also the triangles AEC, DEB.

9. Two circles intersect in A and B. Through A two chords, CAD, EAF, are drawn, terminated both ways by the circumferences. Show that the triangles BCD, BEF are equiangular.

10. Two circles intersect in A and B. Through A two chords, CAD, EAF, are drawn, terminated both ways by the circumferences. Show that CE, DF subtend equal angles at B.

11. The base and vertical angle of a triangle are given; find the locus of the intersection of perpendiculars from the ends of the base on the opposite sides.

12. The base and vertical angle of a triangle are given; find the locus of the intersection of the bisectors of the internal angles at the base of the triangle.

13. ABC is a circle, BC a fixed chord in it, and A any point in the segment BAC. BA is produced to D, so that AD = AC, and DC is joined. Show that D lies on the circumference of a fixed circle through B and C.

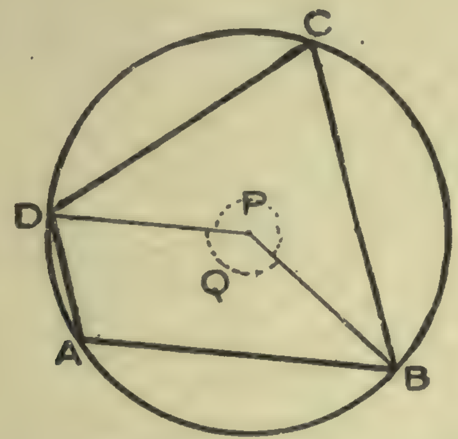
14. Hence show in the figure of the preceding exercise that the sum of the sides BA, AC of the varying triangle BAC, is a maximum when the point A lies on a line bisecting BC at right angles.

15. If the sum of the squares on two lines be given, their sum is a maximum when the lines are equal.

16. If two straight lines AEB, CED, in a circle, intersect at E, show that the angles subtended by AC and BD at the centre are together double of the angle AEC.

PROPOSITION VII. THEOREM.

The opposite angles of a quadrilateral inscribed in a circle are together equal to two right angles.



Let **ABCD** be a quadrilateral inscribed in the circle **ABC**.

Then its opposite $\angle$s *are together equal to two rt.* $\angle$s.

Join **B** and **D** to the centre of the circle; and represent the angles so formed at the centre by **P** and **Q**.

Then $\angle$**A** is one-half of $\angle$**P**,

and $\angle$**C** is one-half of $\angle$**Q**;

$$\therefore \angle A + \angle C = \tfrac{1}{2}(\angle P + \angle Q),$$

$$= \tfrac{1}{2} \text{ of four rt. } \angle\text{s},$$

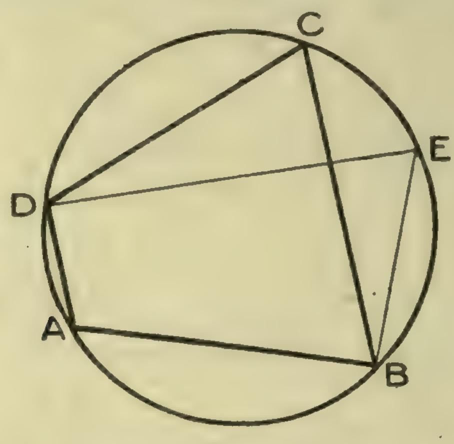
$$= \text{two rt. } \angle\text{s}.$$

Similarly it may be shown that

$$\angle B + \angle D = 2 \text{ rt. } \angle\text{s}.$$

The converse of this proposition is true,—

If two opposite angles of a quadrilateral be together equal to two right angles, the circle which passes through three of its angular points also passes through the fourth.



Let **ABCD** be a quadrilateral, and let the ∠s **A** and **C** be together equal to two rt. ∠s.

Then the circle through **A, B,** *and* **D,** *also passes through* **C.**

Take any point **E** on the arc of this circle cut off by **BD,** and on the same side of **BD** that **C** is.

Join **DE, EB.**

Then ∠**A** + ∠**C** = 2 rt. ∠s. (Hyp.)

Also ∠**A** + ∠**E** = 2 rt. ∠s. (IV., 7.)

Hence ∠**C** = ∠**E,** and ∴, by converse of Prop. VI., **C** lies on the circle through **D, E** and **B,** which is the circle through **A, B** and **D.**

Exercises.

1. If a triangle be inscribed in a circle, show that the sum of the angles in the three segments exterior to the triangle is equal to four right angles.

2. If one side of a quadrilateral inscribed in a circle be produced, show that the exterior angle so formed is equal to the interior opposite angle of the quadrilateral.

✓ 3. Show that no parallelogram except a rectangle can be inscribed in a circle.

✓ 4. A quadrilateral is inscribed in a circle; show that the sum of the angles in the four segments of the circle exterior to the quadrilateral is equal to six right angles.

✓ 5. ABC is an isosceles triangle, and DE is drawn parallel to the base BC, meeting the sides in D and E; show that the points D, B, C, E lie on a circle.

✓ 6. The straight lines which bisect any angle of a quadrilateral inscribed in a circle and the opposite exterior angle, meet on the circumference of the circle.

✓ 7. ABCD is a quadrilateral inscribed in a circle, and the sides DA, CB produced meet in E; show that the triangles EAB, ECD are equiangular.

✓ 8. Hence show that EB.EC = EA.ED.

✓ 9. ABC is a triangle, and in AB, AC points D, E are taken, such that

$$\frac{AB}{AE} = \frac{AC}{AD};$$

show that the points B, C, E, D lie on a circle.

10. Through a given point draw a straight line which shall cut off a cyclic quadrilateral from a given triangle.

✓ 11. ABC is a triangle, and a circle is described to pass through B and C, and to cut AB, AC in D and E. Prove that as the circle through B and C varies, the line DE remains parallel to itself.

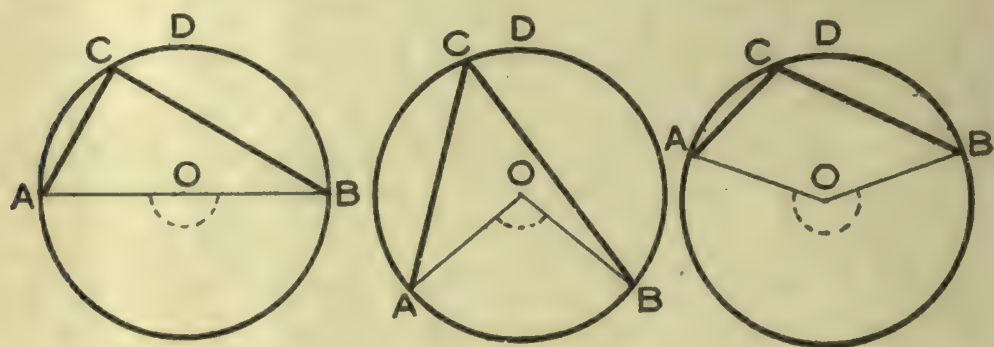
✓ 12. An equilateral triangle is inscribed in a circle; show that the angle subtended at any point on the circumference by one of the sides, is twice the angle subtended at that point by either of the two other sides.

13. If the exterior angles of any quadrilateral be bisected by four straight lines, the quadrilateral formed by these straight lines is cyclic, *i.e.*, a circle may be described about it.

14. From any point P on the circumference of a circle, perpendiculars PA, PB are drawn to two fixed diameters. Show that AB is constant in length.

PROPOSITION VIII. THEOREM.

The angle in a semicircle is a right angle; the angle in a segment greater than a semicircle is less than a right angle; and the angle in a segment less than a semicircle is greater than a right angle.



Let **ABC** be a circle, and **ACB** an $\angle$ in the segment **ADB.**

Then the $\angle$ **ACB** *is equal to, less than, or greater than a rt.* $\angle$, *according as the segment* **ADB** *is a semicircle, or greater than, or less than a semicircle.*

Let **O** be the centre of the circle.

Join **AO, OB.**

Then the $\angle$**AOB** at the centre is $\begin{array}{c}=\\<\\>\end{array}$ 2 rt. $\angle$s,

according as the segment **ADB** is $\begin{array}{c}=\\>\\<\end{array}$ a semicircle;

$\therefore$ the $\angle$**ACB** at the circumference is $\begin{array}{c}=\\<\\>\end{array}$ 1 rt. $\angle$,

according as the segment **ADB** is $\begin{array}{c}=\\>\\<\end{array}$ a semicircle.

Exercises.

1. Right-angled triangles are described on the same hypotenuse; show that the angular points opposite the hypotenuse all lie on a circle described on the hypotenuse as diameter.

2. The circles described on the sides of any triangle as diameters, intersect on the base.

3. AOB, COD are two diameters of a circle, perpendicular to each other. If P be any point on the circumference, prove that PC, PD are the internal and external bisectors of the angle APB.

4. A straight rod of fixed length slides between two straight rulers at right angles to one another. What is the locus of the middle point of the rod?

5. Two circles intersect at A and B, and through A diameters AC, AD are drawn. Show that C, B, D are in the same straight line.

6. Find the locus of the middle points of chords of a circle drawn through a fixed point.

7. Through one of the points of intersection of two circles draw a chord of one circle which shall be bisected by the other circle.

8. Three circles are described, each passing through two of the angular points of a triangle and the intersection of the bisectors of the angles. Show that the centres of the circles all fall without the triangle.

---

9. Three equal circles pass through O, and intersect again, two and two, in A, B and C. P, Q and R are the centres of the circles through B and C, C and A, A and B respectively. Show that ROQA is a rhombus, and hence that PQ is equal and parallel to AB.

Also show that O is the orthocentre of the triangle ABC.

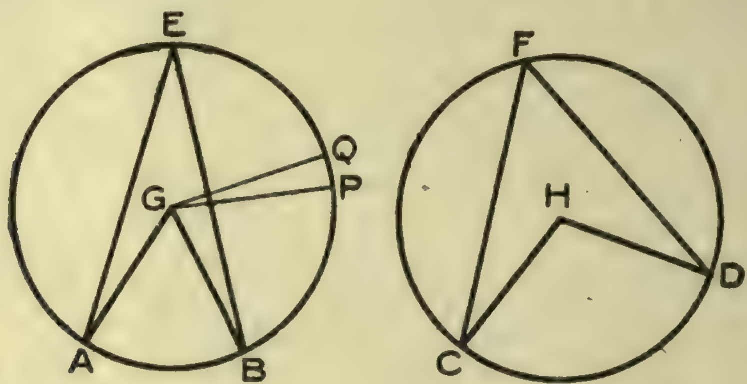
10. A quadrilateral ABCO has the vertex O at the centre of a circle, and the vertices A, B, C on the circumference. Show that $\angle B = \angle A + \angle C$.

11. ABCD is a cyclic quadrilateral. BA, CD, produced, meet in P; and AD, BC, produced, meet in Q. About the triangles ADP, CDQ circles are described; show that they intersect on PQ.

12. The bisectors of the angles formed by producing opposite pairs of sides of a cyclic quadrilateral are at right angles.

PROPOSITION IX. THEOREM.

In equal circles angles, whether at the centres or at the circumferences, have the same ratio as the arcs on which they stand: so also have the sectors.



Let **ABE, CDF** be equal circles, and let **AGB, CHD** be $\angle$s at the centres, and **AEB, CFD** $\angle$s at the circumferences.

$$\textit{Then } \angle \mathbf{AGB} : \angle \mathbf{CHD} = \textit{arc } \mathbf{AB} : \textit{arc } \mathbf{CD};$$

$$\angle \mathbf{AEB} : \angle \mathbf{CFD} = \textit{arc } \mathbf{AB} : \textit{arc } \mathbf{CD};$$

$$\textit{sector } \mathbf{AGB} : \textit{sector } \mathbf{CHD} = \textit{arc } \mathbf{AB} : \textit{arc } \mathbf{CD}.$$

Let **PQ** be an arc which is contained m times in **AB**, and n times in **CD**.

Join **GP, GQ.**

Suppose the arc **AB** divided into m arcs, each equal to **PQ**, and let the points of division be joined to **G.**

Then each of the $\angle$s so formed at **G** will be equal to the $\angle$**PGQ**; and each of the sectors will be equal to the sector **PGQ.** Introductory sec. 15

Hence the $\angle$**AGB** contains the $\angle$**PGQ** m times; and the sector **AGB** contains the sector **PGQ** m times.

Similarly the $\angle$**CHD** contains the $\angle$**PGQ** n times; and the sector **CHD** contains the sector **PGQ** n times.

$$\therefore \angle \mathbf{AGB} : \angle \mathbf{CHD} = m : n = \text{arc } \mathbf{AB} : \text{arc } \mathbf{CD};$$

and sector **AGB** : sector **CHD** $= m : n =$ arc **AB** : arc **CD**.

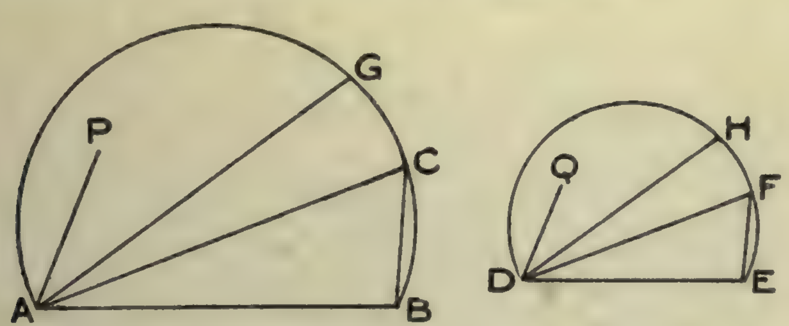
But $\angle$ **AEB** at circumference is half of $\angle$ **AGB** at centre; and $\angle$ **CFD** is likewise half of $\angle$ **CHD**;

$$\therefore \angle \mathbf{AEB} : \angle \mathbf{CFD} = \text{arc } \mathbf{AB} : \text{arc } \mathbf{CD}.$$

NOTE. The Proposition is demonstrated of equal circles and not of the same circle, rather from the convenient figure which is thus obtained. It is manifestly true of the same circle, and indeed it is with the same circle that the Proposition is usually associated.

That the arc is proportional to the angle it subtends at the centre, is our justification for taking arcs to measure angles.

SIMILAR SEGMENTS. Suppose the segments **ACB, DFE** contain equal $\angle$s; and let **AB** : **DE** $= m : n$.



Construct the equal $\angle$s **BAC, EDF.** Join **BC, EF.** Then since the $\angle$s at **C** and **F** are equal, the Δs **CAB, FDE** are equiangular, and

$$\mathbf{AC} : \mathbf{DF} = \mathbf{AB} : \mathbf{DE} = m : n.$$

Similarly, if **AG, DH** make equal $\angle$s with **AB, DE** respectively, then

$$\mathbf{AG} : \mathbf{DH} = m : n.$$

And in general any chords drawn from **A** and **D,** making equal angles with **AB** and **DE** respectively, are in this constant ratio $m : n$.

Such segments are called **similar segments.** It is usual to define similar segments of circles as those which contain equal angles, but it is rather from the property we have just demonstrated that such segments are thought of as *similar;* namely, from the constancy of the ratio of corresponding radius-vectors.

The points **A** and **D** may be called **corresponding points.**

An infinite number of such corresponding points exists.

For take *any* point **P**. Join **AP**. Make the $\angle$**EDQ** $=\angle$**BAP**, and take **DQ** of such length that **AP** : **DQ** $= m : n$. Then it is easy to show that,—

$$\mathbf{PB} : \mathbf{QE} = m : n,$$
$$\mathbf{PC} : \mathbf{QF} = m : n,$$
$$\mathbf{PG} : \mathbf{QH} = m : n,$$

and what was true of **A** and **D** is also true of **P** and **Q**; namely, that corresponding radius-vectors from **P** and **Q** are in this constant ratio $m : n$; *i.e.*, **P** and **Q** are also corresponding points.

Exercises.

1. Show that similar segments of circles whose chords are equal, are congruent.

2. If in two circles equal chords subtend equal angles at the circumferences, the circles are equal.

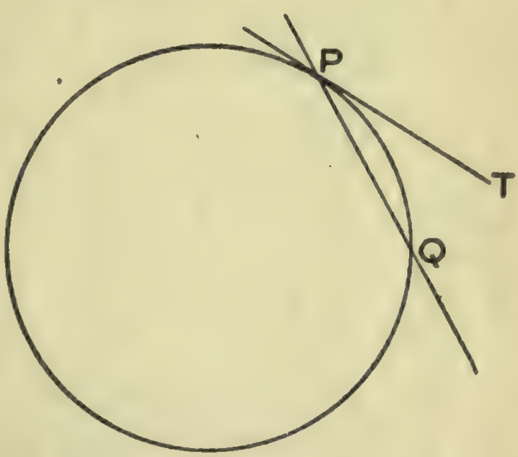
   Show also that if the angles at the circumferences be supplementary, the circles are equal.

3. AB, AC are the equal sides of an isosceles triangle ABC, and D is any point in BC. Circles are described about the triangles ABD, ACD. Show that these circles are equal.

4. **By drawing lines from A and D, making equal angles with AB, DE respectively, in the similar segments AGB, DHE, show how to describe in these segments two similar rectilineal figures. (Page 121.)**

Tangents.

DEFINITION OF TANGENT. **If one of the two points in which a secant cuts the circumference of a circle move up to the other, the ultimate position of the secant, when the points coincide, is the tangent to the circle at the point of coincidence.**

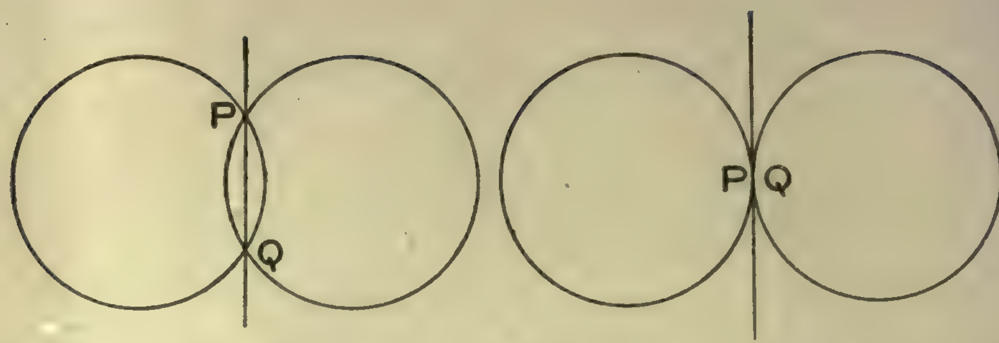


Thus, if the point Q of the secant PQ move *along the circumference* until it becomes indefinitely near to P, the ultimate position of PQ is the tangent at P, as PT. P is called the **point of contact** of the tangent.

In this motion of Q, it is not supposed actually to become the same point as P, but to stop when it is the next point; *i.e.*, a tangent is a straight line passing through two "consecutive points" on a circle.

A tangent and a secant, then, both meet the circle in two points, but in the case of the tangent the two points are indefinitely close to one another. Hence a tangent could not meet the circle at any point other than the point of contact, for it has met it there in two points.

In like manner two circles are said to **touch one another,** or to be **tangential,** when their points of section move up to, and become indefinitely close to one another.



Thus if one, or both, of the circles move so that **P** and **Q** become consecutive points, the circles are said to touch one another, at **P.**

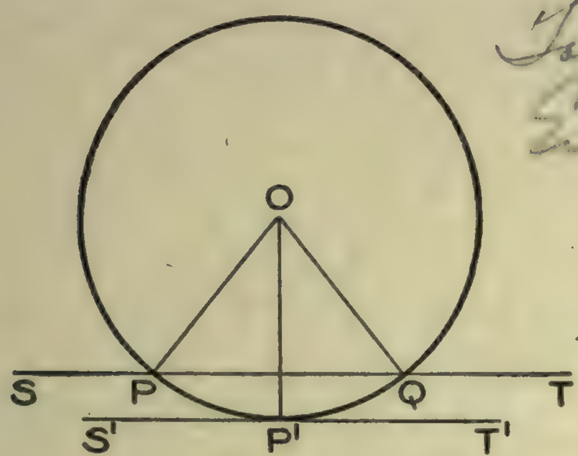
Evidently the secant **PQ** ultimately passes through two consecutive points on each circle, and therefore is a tangent to each circle. So that **if two circles touch they have a common tangent at the point of contact.**

Also, since two circles can meet one another at only two points, **two circles which touch cannot meet one another at any point other than the point of contact,** for they meet there in two points, though these points are indefinitely close to one another.

Another definition of a tangent to a circle:— A straight line is said to touch a circle, or be a tangent to it when it meets the circle but, being produced, does not cut it.

PROPOSITION X. THEOREM.

A tangent to a circle is at right angles to the radius drawn to the point of contact.



To Mackay's Euclid III, 16

Let **P, Q** be two points on the circumference of a circle whose centre is **O**.

Join **PQ**, and produce it both ways to **S** and **T**.

Join **OP, OQ**.

Since **OP** = **OQ**;

∴ ∠**OPT** = ∠**OQS**.

Let now **P** and **Q** move up indefinitely close to one another. Then **OP, OQ** coincide in, say, **OP′**, and **ST** becomes the tangent at **P′**.

Also the ∠s **OPT, OQS** become the ∠s **OP′T′, OP′S′**;

and these are adjacent ∠s;

∴ they are rt. ∠s.

Hence the tangent **S′P′T′** is at rt. ∠s to **OP′**, the radius to the point of contact.

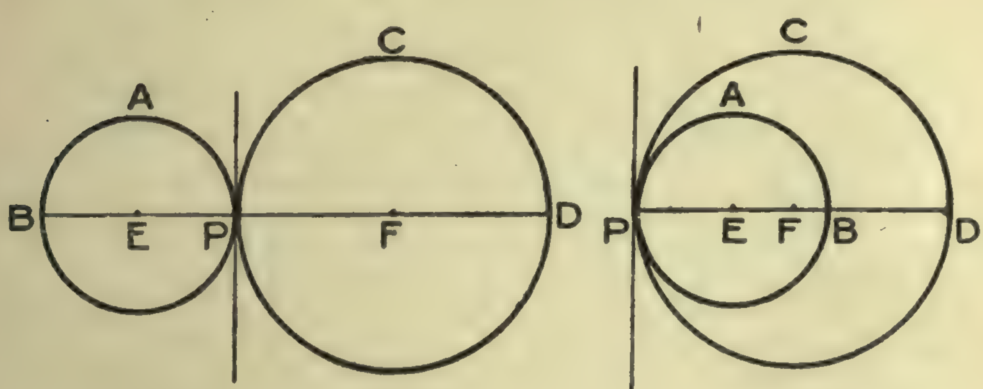
COROLLARY 1. Since only one straight line can be drawn at right angles to a given straight line at a given point, **only one tangent to a circle can be drawn at a given point.**

COROLLARY 2. For the same reason **the perpendicular to a tangent at its point of contact, passes through the centre.**

COROLLARY 3. Since only one straight line can be drawn perpendicular to a given straight line from a point without it, **the line drawn from the centre perpendicular to a tangent passes through the point of contact.**

PROPOSITION XI. THEOREM.

If two circles touch, the straight line joining their centres, produced if necessary, passes through the point of contact.



Let **PAB**, **PCD** be two circles touching at **P**; and let **E**, **F** be their centres.

Then **EF**, *produced if necessary, passes through* **P.**

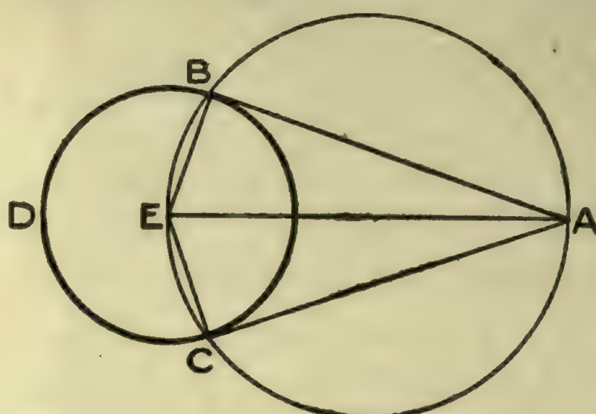
Since the circles touch at **P**, they have at **P** a common tangent. (Page 182.)

Also the radii **PE**, **PF** are both ⊥r to this common tangent at **P**; (IV., 10.)

∴ **E**, **P**, **F** are in the same st. line.

PROPOSITION XII. PROBLEM.

To draw a tangent to a given circle from a given point.



(i) If the given point be within the given circle, since a line through a point within a circle cuts the circumference in two points necessarily *at a finite distance* from one another, no tangent to the circle from the given point can be drawn.

(ii) If the given point be on the circumference, the straight line drawn from the point at right angles to the radius to the point, is the required tangent. (IV., 10.)

(iii) Let **A** be the given point without the given circle **BCD**.

It is required to draw from **A** *a tangent to* **BCD**.

Find **E**, the centre of **BCD**.

Join **AE**.

On **AE** as diameter, describe a circle cutting **BCD** at **B** and **C**.

Join **AB**, **AC**.

Then **AB**, **AC** are tangents to the circle **BCD**.

Because **ABE**, **ACE** are semicircles;

∴ the ∠s **ABE**, **ACE** are rt. ∠s; (IV., 8.)

and **AB**, **AC** are tangents to the circle **BCD**. (IV., 10.)

COROLLARY. Since (IV., In., § 11) **AE**, through the centres of the circles, is an axis of symmetry for both, and **B**, **C** are "images" of each other, therefore if the figure be folded about **AE**, the lines **AB** and **AC** coincide and are equal. That is, two tangents drawn from an external point to a circle are equal. Use I, 8, Cor

Exercises.

✓ 1. Show that all equal chords in a given circle touch a fixed concentric circle.

✓ 2. Through a given point outside a given circle, draw a straight line, such that the part of it intercepted by the circle shall have a given length, not greater than the diameter of the circle.

✓ 3. Through a given point outside a given circle, draw a straight line, such that the part of it intercepted by the circle may subtend at the centre an angle of 60°.

✓ 4. In two concentric circles any chord of the outer circle which touches the inner, is bisected at the point of contact.

✓ 5. Draw a tangent to a given circle parallel to a given straight line.

✓ 6. Draw a tangent to a given circle, making a given angle with a given straight line.

7. If two circles are concentric, all tangents drawn from points on the circumference of the outer to the inner are equal.

8. Find the locus of the centres of all circles which touch each of two parallel lines. ✗

9. In any quadrilateral described about a circle, the sum of one pair of opposite sides is equal to the sum of the other pair.

10. Describe a circle of given radius to touch each of two given straight lines.

11. If a parallelogram can be described about a circle, it must be equilateral.

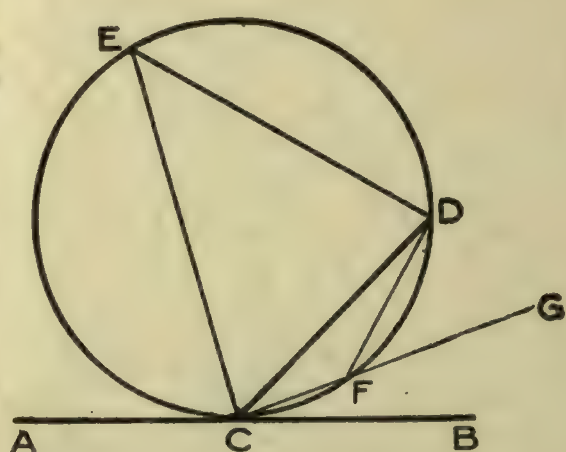
12. If a quadrilateral be described about a circle, the sum of the angles subtended at the centre by each pair of opposite sides is equal to two right angles.

13. Find the locus of the extremities of tangents of fixed length drawn to a given circle. ✗

14. In the diameter of a circle produced, determine a point such that the two tangents drawn from it may contain a given angle.

PROPOSITION XIII. THEOREM.

If a straight line touch a circle, and from the point of contact a chord be drawn, the angles between the chord and the tangent are equal to the angles in the alternate segments of the circle.



Let **ACB** be a tangent to the circle **CDE** at the pt. **C**, and let the chord **CD** be drawn, dividing the circle into segments in which are the ∠s **DFC**, **DEC**.

Then ∠**DCB** = ∠**DEC**, *and* ∠**DCA** = ∠**DFC**.

Produce **CF** to **G**.

Then ∠s **DEC**, **DFC** together = 2 rt. ∠s.

Also ∠s **DFG**, **DFC** together = 2 rt. ∠s;

∴ ∠**DFG** = ∠**DEC**.

Let now **F** move down to **C**; then **CG** coincides with **CB**, and ∠**DFG** coincides with ∠**DCB** and is equal to it;

∴ ∠**DCB** = ∠**DEC**.

In like manner we may prove

∠**DCA** = ∠**DFC**.

Exercises.

1. If from a point A without a circle, a tangent AB and a secant ACD be drawn, show that the triangles ACB, ABD are equiangular.

Hence show that $AB^2 = AC.AD$.

2. Use the preceding exercise to show that tangents drawn to a circle from an external point, are equal.

3. State and prove the converse of Prop. xiii.

4. If two circles touch at P, and chords APB, CPD be drawn through the point of contact, then AC, BD are parallel.

5. If a triangle be inscribed in a circle, and tangents to the circle be drawn at the angular points, the angles of the triangle so formed are the supplements of twice the angles of the former triangle.

6. If two circles touch one another, and through the point of contact a straight line be drawn, the tangents at its ends are parallel to one another.

7. Two circles touch internally, and a chord of the greater touches the less. Show that this chord is divided at its point of contact into segments which subtend equal angles at the point where the two circles touch.

---

8. Of all triangles inscribed in a given circle, the equilateral triangle has the maximum perimeter.

9. Of all triangles having the same base and vertical angle, the sum of the sides is a maximum when the triangle is isosceles.

10. X, Y are any two points on the circumferences of two segments on the same straight line AB, and on the same side of it; the angles XAY, YBX are bisected by two straight lines meeting in Z. Show that the angle AZB is constant, and equal to $\frac{1}{2}(X+Y)$.

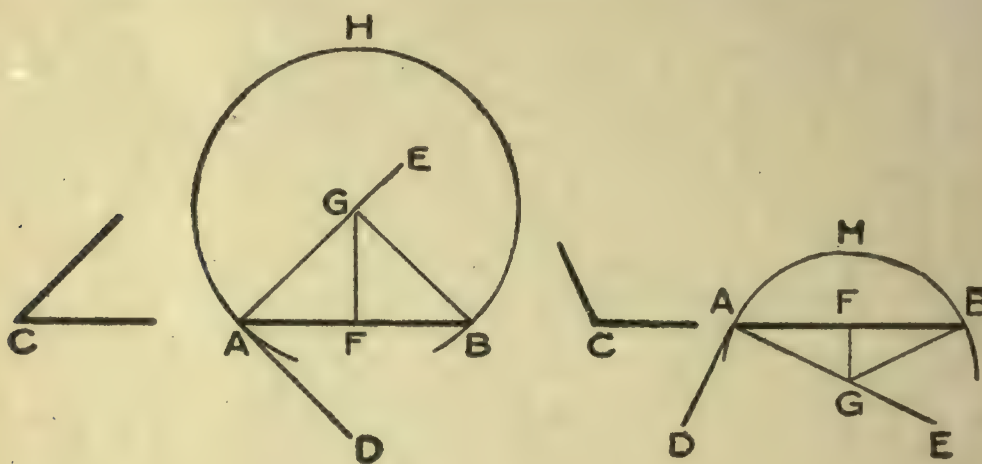
11. ACB is a fixed chord passing through C, the point of intersection of two circles APC, QBC, and PCQ any other chord of the circles passing through C. Show that AP, BQ when produced meet at a constant angle.

12. Describe a circle which shall pass through a given point, and touch a given straight line at a given point.

13. Describe, when possible, a circle of given radius, and touching a given straight line and a given circle.

PROPOSITION XIV. PROBLEM.

On a given straight line to construct a segment of a circle, containing an angle equal to a given angle.



Let **AB** be the given st. line, and **C** the given angle.

It is required to describe on **AB** *a segment of a circle containing an* ∠ *equal to* **C**.

At **A** make ∠**BAD** = ∠**C**.

Draw **AE** ⊥r to **AD**.

Bisect **AB** at **F**.

From **F** draw **FG** ⊥r to **AB**, to meet **AE** in **G**.

Join **BG**.

Then **GA** = **GB**. (Loci, Ex. 2.)

Hence we can describe with centre **G** and radius **GA**, an arc of a circle passing through **B**.

Let it be described on side of **AB** remote from **D**.

Then because **AD** is at rt. ∠s to **AG**,

AD is a tangent to the circle.

Hence ∠ in segment **AHB** = ∠**BAD** = ∠**C**;

and segment required has been described.

Exercises.

1. Find a point O within a triangle ABC, such that the angles AOB, BOC, COA are equal to one another.

2. Construct the locus of the vertices of all triangles on a given base and having a given vertical angle.

3. Construct a triangle, having given the base, the vertical angle, and one other side.

4. Construct a triangle, having given the base, the vertical angle, and the point at which the perpendicular from the vertex meets the base.

5. Construct a triangle, having given the base, the vertical angle, and the point at which the base is cut by the bisector of the vertical angle

6. Construct a triangle, having given the base, the vertical angle, and the distance from the vertex to the middle point of the base.

7. Construct a triangle, having given the base, the vertical angle, and the sum of the remaining sides.

---

8. Four circular coins of different sizes are lying on a table, each touching two, and only two, of the others. Show that the four points of contact lie on a circle.

9. A tangent to a circle is drawn parallel to a chord; show that the point of contact bisects the arc cut off by the chord.

10. If two circles touch, and a straight line be drawn through the point of contact, it divides the circles into segments that are similar in pairs.

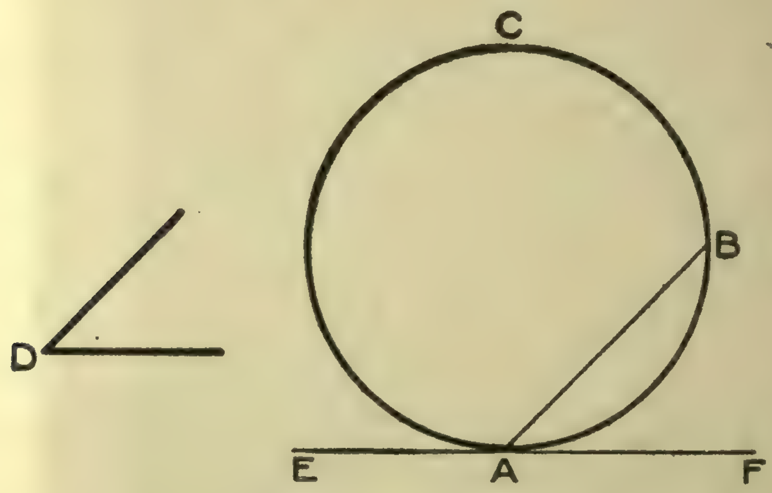
11. There are two concentric circles, and a straight line ABC cuts one of them in A and the other in B and C. Show that the tangents at B and C intersect the tangent at A, at points equally distant from the common centre.

12. A circle passes through a fixed point P, and cuts a fixed straight line at a point Q, so that the tangent at Q makes a constant angle with PQ. Show that the circle meets the fixed straight line in a fixed point.

13. Given the vertical angle, the perimeter, and the altitude of a triangle, construct it. (On a st. line equal to perimeter construct a segment of a circle with angle equal to vert. angle + $\frac{1}{2}$ sum of other angles; and draw a st. line parallel to base at distance from it equal to given altitude.)

PROPOSITION XV. PROBLEM.

From a given circle to cut off a segment containing an angle equal to a given angle.



Let **ABC** be the given circle, and **D** the given ∠.

It is required to cut off from the circle **ABC** *a segment containing an* ∠ *equal to* **D.**

Draw **EAF**, the tangent at **A.**

Draw the chord **AB**, making the ∠**FAB** = ∠**D.**

Then **ACB** is segment required.

For ∠**D** = ∠**FAB** (const.)

= ∠ in alternate segment **ACB.** (IV., 13.)

Exercises.

1. Make the construction required in the Proposition without drawing a tangent.

2. The chord of a segment of a given circle is produced. On the whole line so produced describe, with the simplest possible construction, a segment of a circle containing an angle equal to the angle in the segment of the given circle.

3. Through a given point without a circle draw a straight line that will cut off a segment containing an angle equal to a given angle.

4. If a point be within a circle, what is the greatest and what the least angle in a segment cut off by a chord through it?

5. Draw a line parallel to a given line and cutting off from a given circle a segment containing an angle equal to a given angle.

6. By using an angle double the given angle, make the construction required in Prop. xv.

7. In a given circle describe an isosceles triangle equiangular to a given isosceles triangle.

---

8. From two given circles cut off similar segments.

In these segments describe two similar quadrilaterals (Ex. 4, p. 180); and show that the ratio between corresponding sides of the quadrilaterals is equal to the ratio between the radii of the circles.

9. The circumference of a circle P passes through the centre of a circle Q. Show that the tangents to Q, at the points of intersection, meet on the circumference of P.

10. A straight line AB of given length, moves with its ends resting on two fixed lines which intersect in O. Show that the locus of the centre of the circle through the points O, A and B is a circle.

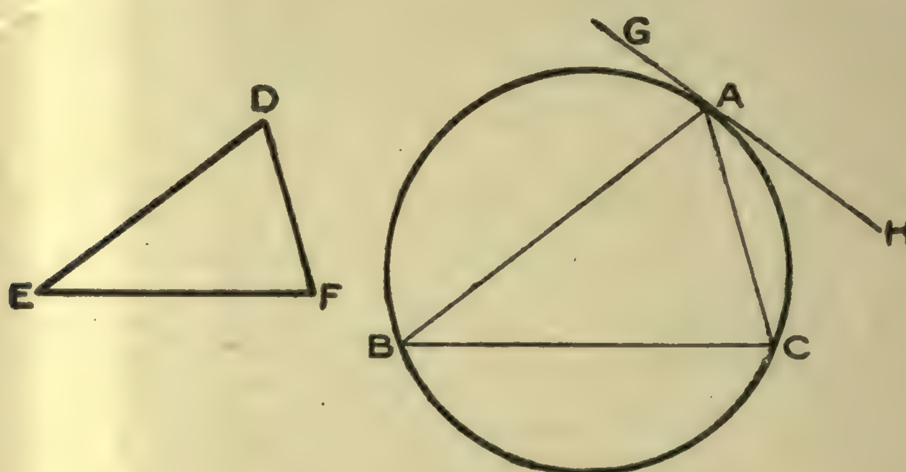
11. Tangents to a circle, TP, TQ, intercept between them the arc QOP; and from O, perpendiculars OL, OM, ON are drawn to TQ, TP, PQ. Show that ON is a mean proportional between OL and OM.

12. A is a fixed point, and from it lines ABC, ADE, are drawn to meet two fixed lines in B and C, D and E. If circles be described about the triangles ABD, ACE, show that they intersect at a constant angle.

13. Two circles BPQ, CPQ are tangential to two fixed lines AB, AC at the fixed points B and C. Show that, as the circles vary, if P be on the circumference of a circle through B and C, then Q is also on the circumference of a circle through B and C.

PROPOSITION XVI. PROBLEM.

In a given circle to inscribe a triangle equiangular to a given triangle.



Let **ABC** be the given circle, and **DEF** the given $\triangle$.

It is required to inscribe in the circle **ABC** *a* $\triangle$ *equiangular to* **DEF**.

Draw a tangent to the circle at any point **A**.

Draw a chord **AB**, making $\angle$ **GAB** = $\angle$ **F**,

and a chord **AC**, making $\angle$ **HAC** = $\angle$ **E**.

Join **BC**.

Then the $\triangle$**ABC** is equiangular to $\triangle$**DEF**.

For $\angle$ **F** = $\angle$ **GAB** = $\angle$ **C** in alt. segment;

and $\angle$ **E** = $\angle$ **HAC** = $\angle$ **B** in alt. segment,

$\therefore$ remaining $\angle$ **EDF** = remaining $\angle$ **BAC**,

and $\triangle$**ABC** is equiangular to $\triangle$**DEF**.

Exercises.

1. How would you make the construction in the Proposition, that the sides of the triangle ABC may be parallel to those of DEF?

2. How, that the sides of ABC may be perpendicular to the sides of DEF?

3. If a circle be described about the triangle DEF, show that the ratio of corresponding sides in the two triangles is equal to the ratio of the radii of the circles.

4. If another triangle equiangular to DEF be inscribed in the circle ABC, show that it is equal in all respects (congruent) to the triangle ABC.

5. Inscribe an equilateral triangle in a given circle, and show that if tangents to the circle be drawn at the angular points, the triangle so formed is equilateral.

6. In a given circle inscribe a triangle whose sides are parallel to three given straight lines.

7. If an equilateral triangle be inscribed in a circle, show that its area is $\frac{3\sqrt{3}}{4} r^2$, where r is the radius of the circle.

---

8. In a given circle inscribe a triangle ABC, such that the angle A is of given magnitude, and that the sides AB, AC pass through given points D and E respectively. Is problem always possible?

9. Two circles intersect in A and B, and CAD is a straight line terminated both ways by the circumference. Find the position of CAD, that the area of the triangle BCD may be the greatest possible.

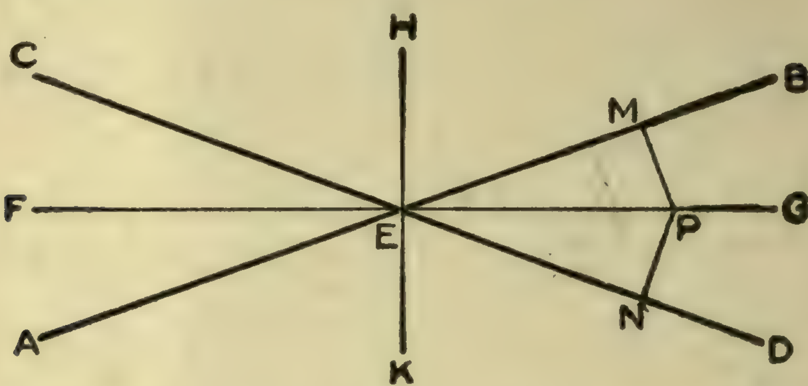
10. O is the orthocentre of the triangle ABC, and the parallelogram OBPC is completed. Show that AP is the diameter of the circle about the triangle ABC.

11. In a given segment, BAC, of a circle, place two lines AB, AC such that their ratio may be equal to a given ratio. (Divide BC in the given ratio in D, and on BD or DC describe a segment containing an angle equal to half of angle in BAC.)

12. Two circles whose centres are A and B touch externally at P, and CPD is drawn meeting the circles in C and D. Show that the triangles APD, CPB are equal in area.

PROPOSITION XVII. PROBLEM.

To find the loci of the centres of circles touching two given straight lines.



Let **AB**, **CD** be the two given st. lines, intersecting at **E**.

It is required to find the loci of the centres of circles touching **AB** *and* **CD**.

Bisect the ∠s **AEC**, **BED** by the st. line **FEG**; and from any pt. **P** on **FEG**, draw **PM**, **PN** at rt. ∠s to **AB**, **CD** respectively.

Then **PM** = **PN**; (Loci, Ex. 1.)

and circle described with centre **P** and radius **PM** will pass through **N**, and touch **AB**, **CD** at **M** and **N**, because the ∠s at **M** and **N** are rt. ∠s.

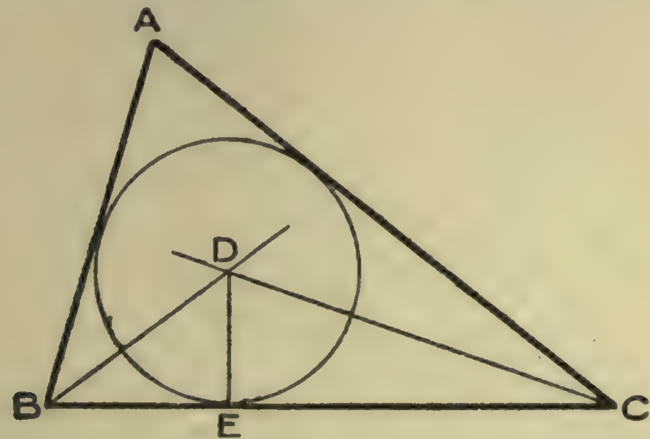
Hence **FEG**, bisecting one of the ∠s between **AB** and **CD**, is a locus of centres of circles touching **AB** and **CD**.

Similarly **HEK**, bisecting the other ∠ between **AB** and **CD**, is a locus of centres of circles touching **AB** and **CD**.

NOTE. A circle is said to be inscribed in a rectilineal figure when the circumference touches all the sides of the figure.

PROPOSITION XVIII. PROBLEM.

To inscribe a circle in a given triangle.



Let **ABC** be the given triangle.

***It is required to inscribe a circle in* ABC.**

Bisect the ∠s **ABC, ACB** by the st. lines **BD, CD,** intersecting at **D.**

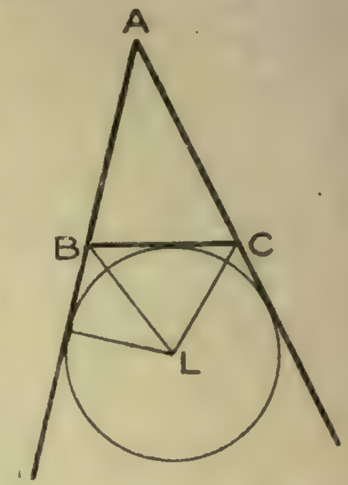
Then **BD** is a locus of centres of circles touching the sides **BA, BC.** (IV., 17.)

And **CD** is a locus of centres of circles touching the sides **CA, CB.**

Hence **D** is the centre of a circle which can be described to touch **AB, BC, CA.**

From **D** draw **DE** ⊥r to **BC,** and with centre **D,** and radius **DE,** describe a circle. It will touch **AB, BC, CA;** and be inscribed in **△ABC.**

If the sides **AB, AC** be produced, and the exterior angles at **B** and **C** be bisected by straight lines **BL, CL** meeting in **L**, then **L** is the centre of a circle which may be described to touch the productions of **AB, AC** and the side of **BC** remote from **A.** Such a circle is called an **escribed** circle. Evidently circles may also be escribed to the sides **AB, AC.** We thus have four circles touching the sides of the triangle. Indeed, the problem of the Proposition stated in its general form is,—**To describe a circle touching three intersecting straight lines,** and we have four different circles as solutions of the problem.



NOTE. A circle is said to be described about a rectilineal figure when the circumference passes through all the angular points of the figure. **To describe a circle about a triangle,** see Loci, Ex. 4.

Exercises.

1. In the figure of the Proposition, show that AD bisects the angle BAC.

2. If A be joined to the centre of the escribed circle touching BC, show that the joining line passes through D.

3. Show that the line joining the centres of any two escribed circles passes through an angle of the triangle.

4. If four circles be described touching three intersecting lines, prove that two centres of circles and a point of intersection of the lines, are always collinear.

5. If r be the radius of the inscribed circle of a triangle whose sides are a, b, and c, prove that

$$\text{area of triangle} = \tfrac{1}{2} r\,(a+b+c).$$

6. Show that all the sides of the triangle subtend obtuse angles at the centre of the inscribed circle, their values being $90° + \frac{1}{2}$ A, $90° + \frac{1}{2}$ B, $90° + \frac{1}{2}$ C.

7. Show that if the base and vertical angle of a triangle be given, the locus of the centre of the inscribed circle is a circular arc.

8. Show how to describe a circle which shall cut off equal chords from the sides of a given triangle.

9. If perpendiculars be drawn from the centre of an escribed circle of a triangle to the sides of the triangle, two of the angles between these perpendiculars are equal to angles of the triangle, and the third is equal to the supplement of the third angle of the triangle.

10. Construct a triangle equiangular to a given triangle, and having a given circle for one of its escribed circles.

11. With the vertices A, B, C, of a triangle as centres, describe three circles, each of which touches the other two.

12. If, in the figure of the Proposition, a triangle be cut off at each angle by a tangent to the circle, the sum of the perimeters of the three triangles so cut off is equal to the perimeter of the original triangle.

13. Without producing two straight lines to meet, find the straight line which would bisect the angle between them.

14. If two sides of a triangle whose perimeter is constant are given in position, prove that the third side rolls on a certain circle.

15. If a triangle be formed by joining the points of contact of the inscribed circle, the angles of the triangle so formed are $\frac{1}{2}$ (A + B); $\frac{1}{2}$ (B + C), $\frac{1}{2}$ (C + A).

Show that this triangle is equiangular to the triangle formed by joining the centres of the escribed circles.

16. If in the Proposition the circle touch the sides AB, AC in H and G, show that the middle point of the arc HG is the centre of the circle inscribed in the triangle AHG.

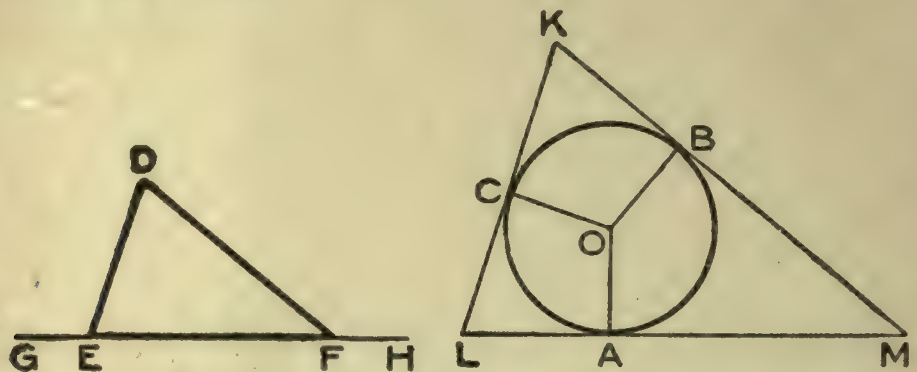
17. If a, b, c be the sides of a triangle, prove that the segments of the sides made by the points of contact of the inscribed circle are

$$\tfrac{1}{2}(b+c-a),\ \tfrac{1}{2}(c+a-b),\ \tfrac{1}{2}(a+b-c).$$

18. Given the base of a triangle, the vertical angle, and the radius of the inscribed circle; construct the triangle.

PROPOSITION XIX. PROBLEM.

About a given circle to describe a triangle equiangular to a given triangle.



Let **ABC** be the given circle, and **DEF** the given $\triangle$.

It is required to describe about **ABC** *a* $\triangle$ *equiangular to* **DFF**

Produce **EF** both ways to **G** and **H**.

Take **O**, the centre of the circle **ABC**, and draw any radius **OA**.

At **O** in **AO** make the $\angle$ **AOB** $=$ $\angle$ **DFH**, and $\angle$ **AOC** $=$ $\angle$ **DEG**, **B** and **C** being on the circumference.

At **A**, **B** and **C** draw tangents **LM**, **MK** and **KL** to the circle, forming the $\triangle$**KLM**.

Then $\triangle$**KLM** is equiangular to $\triangle$**DEF**.

Because $\angle$s at **A** and **B** are rt. $\angle$s;

$\therefore$ $\angle$s **AOB**, **AMB** together $=$ 2 rt. $\angle$s;

but $\angle$s **DFH**, **DFE** together $=$ 2 rt. $\angle$s;

and $\angle$ **DFH** $=$ $\angle$ **AOB**;

$\therefore$ $\angle$ **AMB** $=$ $\angle$ **DFE**.

Similarly $\angle$ **ALC** $=$ $\angle$ **DEF**;

and $\therefore$ the remaining $\angle$ **K** $=$ remaining $\angle$ **D**;

and $\triangle$**KLM**, described about the circle **ABC**, is equiangular to $\triangle$**DEF**.

Exercises.

1. In the Proposition show that if a circle be also inscribed in the triangle DEF, the ratio of the radii is equal to the ratio of corresponding sides of the triangles.

2. If circles be described about two equiangular triangles, show that the ratio of the radii is equal to the ratio of corresponding sides of the triangles.

3. If another triangle equiangular to DEF be described about the circle ABC, show that the two triangles about the circle are equal in all respects.

4. In the figure of the Proposition, AO is produced to meet the circle again in P, and through P a tangent is drawn meeting KL, KM in Q and R respectively. Show that the triangle KQR is equiangular to DEF.

5. In the Proposition show that the tangents at A, B and C intersect so as to form a triangle about the circle.

6. Given a circle, draw tangents to it so as to form a triangle to which the circle shall be escribed, the triangle so formed being equiangular to a given scalene Δ, and with sum of sides least.

7. About a given circle describe a quadrilateral equiangular to a given quadrilateral.

8. If an equilateral triangle be described about a circle, and the points of contact be joined, the triangle so formed is also equilateral.

9. If the triangle formed by joining the points of contact A, B, C in the Proposition, be equiangular to the triangle KLM, then both triangles must be equilateral.

10. If the centres of three of the circles which touch the sides of a triangle be given, construct the triangle.

11. In the figure of the Proposition, inscribe within the circle a triangle equiangular to KLM, and having its sides parallel to those of KLM.

12. Find the radius of the circle ABC, such that when KLM, equiangular to DEF, is described about it, KLM shall be in area double of DEF.

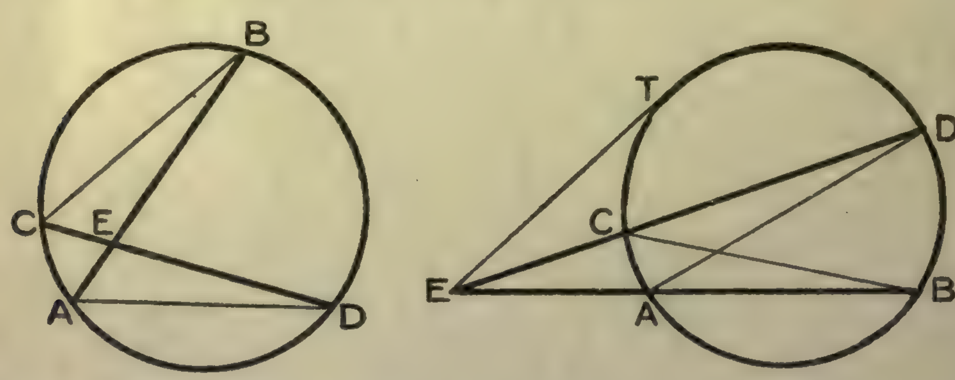
Segments of Intersecting Chords.

A straight line may be divided into segments externally as well as internally. Thus the line **AB** is inter-

nally divided at **C**, giving the segments **AC**, **CB**; and externally divided at **D**, giving the segments **AD**, **DB**.

PROPOSITION XX. THEOREM.

If two chords of a circle intersect, either within or without the circle, the rectangle contained by the segments of one is equal to the rectangle contained by the segments of the other; and if the point of intersection be without the circle, the rectangle contained by the segments of a chord is equal to the square on the tangent from that point.



Let the chords **AB**, **CD** of the circle **ABC** intersect in **E**.

Then the rectangle contained by **AE**, **EB** *is equal to the rectangle contained by* **CE**, **ED**.

In the Δs **EAD**, **ECB**,

$\angle$**D** $=$ $\angle$**B**, since they stand on same arc **AC**;

and $\angle$s **AED**, **CEB** are equal,

being either vertically opposite or coincident;

$\therefore$ Δs **EAD, ECB** are equiangular;

$$\text{and } \frac{\mathbf{AE}}{\mathbf{CE}} = \frac{\mathbf{ED}}{\mathbf{EB}}; \qquad \text{III, 5}$$

that is, $\mathbf{AE} \cdot \mathbf{EB} = \mathbf{CE} \cdot \mathbf{ED}$.

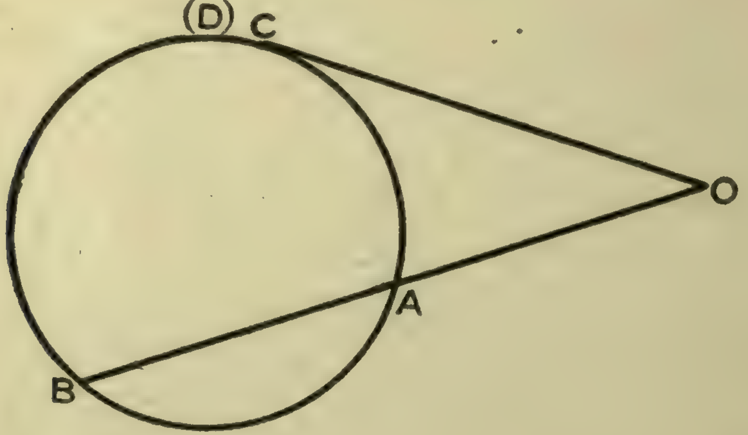
Again, in the second figure, let the secant **ECD** turn about **E**, and become the tangent **ET**, the two points **C** and **D** coinciding in **T**, the segments **CE**, **ED** both becoming **ET**, and the rectangle **CE . ED**, therefore, becoming $\mathbf{ET}^2$.

$$\text{Then } \mathbf{AE} \cdot \mathbf{EB} = \mathbf{CE} \cdot \mathbf{ED}, \text{ always,}$$
$$= \mathbf{ET}^2.$$

The Proposition may be otherwise expressed as follows,—**If a chord of a circle pass through a fixed point, the rectangle contained by the segments of the chord is constant, and is equal to the square on half the chord bisected by the point, if the point be within the circle, and to the square on the tangent from the point, if the point be without the circle.**

PROPOSITION XXI. THEOREM.

If OAB and OC be two straight lines, and OA . OB = OC^2, then OC is a tangent to the circle through the points A, B and C.



Let a circle be described passing through **A, B** and **C**, and suppose **OC** to cut this circle again in **D**.

$$\text{Then } OC^2 = OA \cdot OB, \quad \text{(Hyp.)}$$

$$= OC \cdot OD; \quad \text{(IV., 20.)}$$

$$\therefore OC = OD;$$

that is, the two points **C** and **D** are coincident.

OC therefore can meet the circle only in coincident points, and is a tangent.

This Proposition is the converse of the latter part of Prop. XX.

The point **D** could not fall on the other side of **O**, for then **O** would be within the circle; but it must be external since it does not lie between **A** and **B**.

If **O** lie *between* **A** and **B**, and $\mathbf{OA} \cdot \mathbf{OB} = \mathbf{OC}^2$, then **OC** is half of the chord through **C** and **O**, of the circle through **A**, **B** and **C**.

Exercises.

1. If two straight lines AB, CD intersect at E, and AE.EB = CE.ED, show that the four points A, C, B, D are concyclic.

2. ABC is a triangle right angled at C; and CD is drawn at right angles to AB. Show that $CD^2 = AD.DB$.

3. ABC is a triangle, and BE, CF are drawn perpendicular to AC, AB, and intersecting in O. Show that BO.OE = CO.OF.

4. Two circles intersect in P and Q, and through any point in PQ a chord is drawn in each circle; show that the ends of these chords are concyclic.

5. As an immediate deduction from Prop. xx., show that tangents to a circle from the same point are equal.

6. **If the common chord of two intersecting circles is produced to any point, the tangents to the two circles from this point are equal.**

7. **If the tangents from any point to two intersecting circles be equal, that point must be on the common chord produced.**

Hence find the locus of the point from which equal tangents can be drawn to two given intersecting circles.

This locus is called the Radical Axis. The locus is also a straight line when the circles do not intersect.

8. If the common chord of two intersecting circles be produced to cut a common tangent, it will bisect it.

9. If three circles intersect, the three common chords pass through the same point.

√ 10. A number of circles all pass through two given points A and B, and from a point in AB produced, tangents are drawn to the circles; find the locus of the points of contact.

11. If two straight lines AB, CD, produced, meet in O, and $OA.OB = OC.OD$; show that A, B, C, D are concyclic.

12. If two circles intersect, and through a point on their common chord produced, two secants are drawn, one to each circle; show that the four points of section of the secants with the circles are concyclic.

13. If a straight line be divided into two parts, show that the nearer the point of division is to the middle of the line, the greater will be the rectangle contained by the parts. (Describe circle on line as diameter.)

14. Through a given point without a circle draw, when possible, a straight line cutting the circle so that the part within the circle may equal the part without.

15. If BE, CF are drawn at right angles to the sides AC, AB of the triangle ABC, then $AF.AB = AE.AC$.

16. From a given point as centre, describe a circle cutting a given straight line in two points, so that the rectangle contained by their distances from a fixed point in the straight line may be equal to a given square.

17. Produce a given straight line AB to C, so that the rectangle AC.CB may be equal to a given square.

18. The tangents from a fixed point to a series of intersecting circles are equal to one another. Show that the common chord of each pair of circles passes through this point.

19. Each of three given circles touches the other two; show that the common tangents at the three points of contact will meet in a point.

20. There is a series of circles A, B, C, D, ..., such that B touches A and C, C touches B and D, ... From a point P a secant PQR is drawn, and PQ.PR is constant for all the circles; show that the tangents at the points of contact of the circles all pass through P.

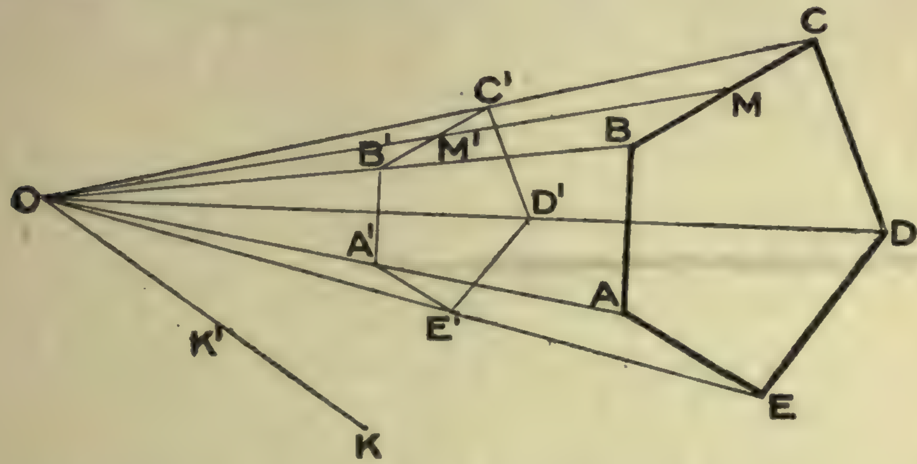
21. Two circles touch one another at P and are cut by a third in the points A, B and C, D respectively. Show that AB and CD intersect on the common tangent at P.

BOOK V.

SIMILAR POLYGONS.

PROPOSITION I. PROBLEM.

To construct a rectilineal figure similar to a given rectilineal figure, and having its sides to those of the given figure in a given ratio.



Let **ABCDE** be the given rectilineal figure, and $m : n$ the given ratio.

It is required to construct a figure similar to **ABCDE,** *and having its sides to the corresponding sides of* **ABCDE** *in the ratio* m : n.

Take any point **O**; and join **OA, OB, OC, OD, OE.**

Take a line **OK′K**; and let **OK′** contain m units of length, and **OK** n units of length.

Divide **OA** at **A′** as **OK** is divided at **K′** (III., 9); so that **OA′ : OA** $= m : n$.

Draw **A′B′** ∥ to **AB**; **B′C′** ∥ to **BC**; **C′D′** ∥ to **CD**; **D′E′** ∥ to **DE.**

Join **A′E′.**

Then **A′B′C′D′E′** is similar to **ABCDE,** and has its sides to those of **ABCDE** in the ratio $m : n$.

Since **A′B′** is parallel to **AB, OA′ : OA = OB′ : OB;**
" **B′C′** " **BC, OB′ : OB = OC′ : OC;**
" **C′D′** " **CD, OC′ : OC = OD′ : OD;**
" **D′E′** " **DE, OD′ : OD = OE′ : OE.**

Hence **OA′ : OA = OE′ : OE**, and ∴ **A′E′** is ‖ to **AE.**

Accordingly the sides of **A′B′C′D′E′** are ‖ to those of **ABCDE**, and the two figures are equiangular.

Again, since the Δs **OA′B′**, **OAB** are similar,

∴ **A′B′ : AB = OA′ : OA** $= m : n$.

Also **OB′ : OB = OA′ : OA** $= m : n$.

Then from similar Δs **OB′C′**, **OBC**,

B′C′ : BC = OB′ : OB $= m : n$.

In like manner we may show that

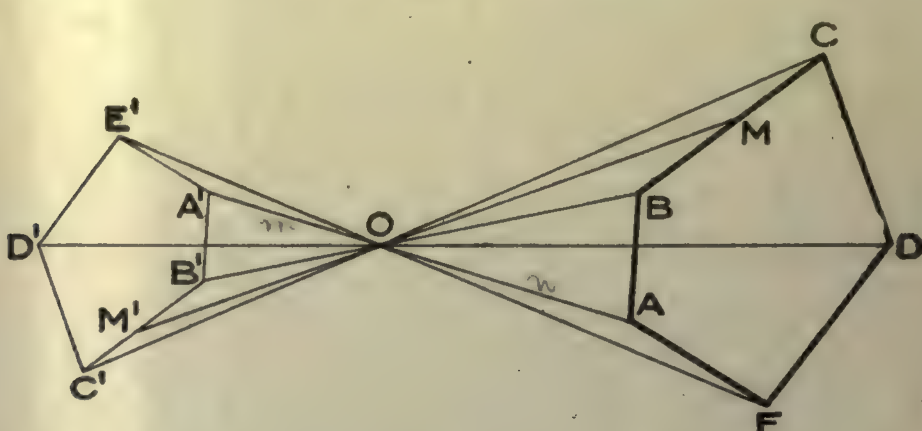
C′D′ : CD $= m : n =$ **D′E′ : DE = E′A′ : EA.**

Hence the figures **A′B′C′D′E′**, **ABCDE** are equiangular, and corresponding sides are in the same ratio, $m : n$.

They are therefore similar, and corresponding sides are in the ratio $m : n$.

The point **O** is called a **centre of similitude**, and in this case is said to be **external.**

If in the construction **AO** had been produced to **A′**, so that **A′O : OA** $= m : n$, and the construction proceeded



with from the point **A′**, the point **O** would be an **internal centre of similitude.** In this latter diagram, if **A′B′C′D′E′** be turned through 180°, it is placed with respect to **ABCDE** as in the former diagram.

If any line **OM′M** (**M′OM**) be drawn through **O**, the points **M′**, **M**, in which it cuts corresponding sides, may be called **corresponding points**.

COR. 1. Evidently, from similar $\triangle$s, **OB′M′**, **OBM**,

$$OM' : OM = OB' : OB = m : n.$$

If any other line **ON′N** be drawn,

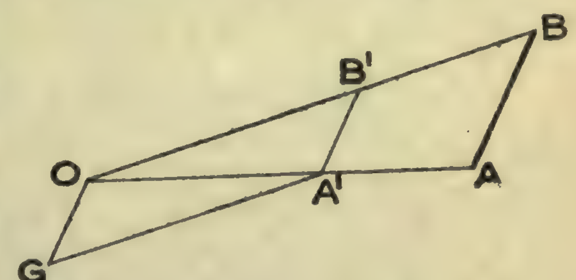
$$ON' : ON = m : n.$$

Hence $\triangle$s **OM′N′**, **OMN** are similar,

and $M'N' : MN = m : n$;

that is, **lines joining pairs of corresponding points are in the ratio** $m : n$.

COR. 2. We may readily describe on a line of given length a polygon similar to another. For between **OA** and **OB** place a line **A′B′** parallel to **AB** and of the given length, as suggested by the annexed diagram. **OG** is drawn of the given length parallel to **AB**, and **GA′** is drawn parallel to **OB**. Starting, then, from **B′**, we proceed as in the Proposition.



NOTE. It is important to observe that not only are such points as **A** and **A′**, **B** and **B′**, . . . , **M** and **M′**, **N** and **N′**, corresponding points, but also that to every point *within* the perimeter of **ABCDE** there exists a corresponding point within the perimeter of **A′B′C′D′E′**. For within the perimeter of **ABCDE** take any point **X**. Join **OX**; and divide **OX** at **X′** so that $OX' : OX = m : n$. Then **X, X′** are corresponding points. If **Y, Y′** be also corresponding points, obtained in the same way, evidently $X'Y' : XY = m : n$, the constant ratio between corresponding distances in the two figures.

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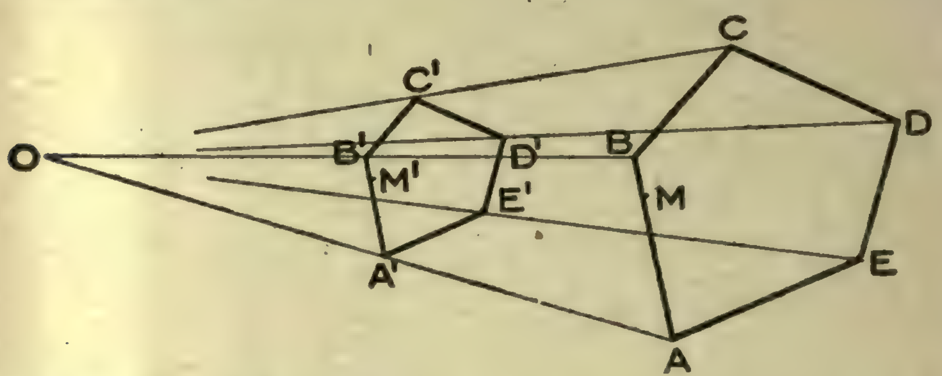
PROPOSITION II. THEOREM.

Two similar polygons may be so placed that the lines joining corresponding angular points are concurrent.



Let **ABCDE, A′B′C′D′E′** be two similar polygons, and let **A′B′C′D′E′** be so placed that **A′B′** is ∥ to **AB**, and **B′C′** ∥ to **BC**.

Then the lines joining **A** *and* **A′**, **B** *and* **B′**, . . ., *are concurrent.*

Since the polygons are equiangular, **C′D′**, **D′E′** and **E′A′** are ∥ to **CD, DE, EA** respectively.

Join **AA′**, **BB′** and let them meet, if necessary when produced, in **O**.

Let $m : n$ be the ratio between corresponding sides.

Then since **A′B′** is ∥ to **AB**,

$$OB' : OB = A'B' : AB = m : n.$$

Also $B'C' : BC = m : n$;

and ∠s **OB′C′**, **OBC** are equal, since **B′C′** is ∥ to **BC**;

∴ △s **OB′C′**, **OBC** are similar. (III., 7.)

Hence the ∠s **B′OC′**, **BOC** are equal,

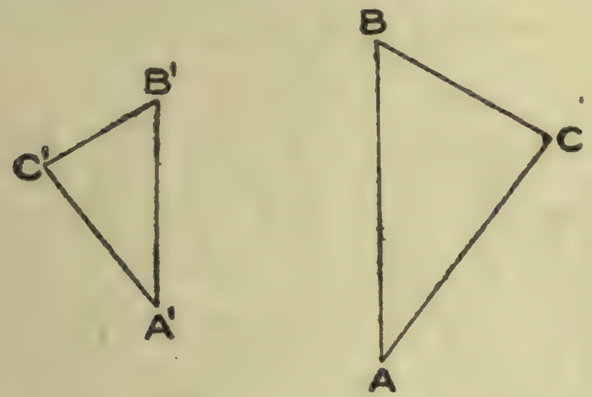
and **OC′**, **OC** are in the same st. line;

that is, **CC′** passes through **O**.

Similarly **DD′** and **EE′** pass through **O**.

COR. If points **M, M′** be taken in any corresponding sides **AB, A′B′**, such that **A′M′ : AM** $= m : n$, evidently the $\triangle$s **OA′M′, OAM** are similar, and **MM′** passes through **O.** Thus **M, M′** are corresponding points, as defined at the close of the preceding proposition.

It is to be observed that we cannot always make the corresponding sides of two similar figures parallel by turning one of them through an angle in the plane in which they both are. Thus the two triangles **ABC,**



A′B′C′, supposed similar, can only be made to have corresponding sides parallel by lifting one from the plane of the paper, turning it over, and then adjusting it.

If two similar figures are such that without taking either from the plane of the paper they can be "similarly placed," they are said to be **directly similar;** if they are such that they can be similarly placed only after one is lifted from the plane of the paper, turned over, and then adjusted, they are said to be **inversely similar.**

Exercises.

1. Having constructed two similar polygons, find pairs of corresponding points not on the perimeters of the polygons. (See p. 180.)

2. Show that the lines joining such corresponding points pass through the centre of similitude.

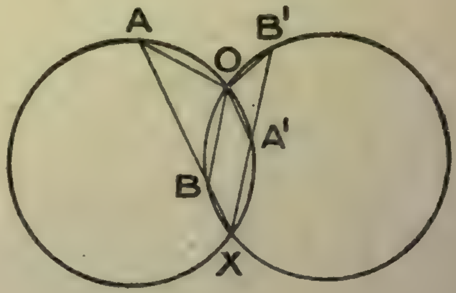
(Observe that a centre of similitude is where corresponding points coincide.)

3. ABCDE being a polygon, and A′B′ a line in the same straight line with AB, construct on A′B′ a polygon similar to ABCDE, first finding a point which shall be a centre of similitude for the two figures.

4. Construct an irregular pentagon ABCDE, and on AB construct an equal and inversely similar pentagon BAC′D′E′. (The construction will be facilitated by drawing a perpendicular to AB through its middle point, and parallels to AB through C, D and E. The intersections of BC, etc., with the perpendicular, determine the directions of AC′, etc.; and the parallels fix their lengths.)

5. Construct a pentagon inversely similar to, and with linear dimensions half those of, the irregular pentagon ABCDE.

6. On AB, A′B′, two lines which are not parallel, construct two directly similar triangles OAB, OA′B′. (Let AB, B′A′ meet in X; and about AA′X, BB′X describe circles intersecting again in O. Then OAB, OA′B′ are equiangular and similar; and OA : OA′ = OB : OB′ = AB : A′B′ = $m : n$.)



7. ABCDE, A′B′C′D′E′ are two directly similar polygons whose corresponding sides are not parallel. Find a point O such that

$$OA : OA' = OB : OB' = OC : OC' = \ldots = m : n,$$

the ratio between corresponding sides of the polygons.

(O is said in this case to be the centre of similitude of the two polygons which are not "similarly placed.")

8. In the preceding exercise, how many centres of similitude are there for the pentagons?

(Observe that when corresponding sides become parallel, the centres of similitude move into coincidence.)

9. Describe two regular pentagons, place them similarly, and find the centre of similitude. Make the same construction for two regular hexagons.

10. When the number of sides of two regular polygons, similarly placed, is the same and even, how many centres of similitude are there?

11. In two concentric circles, describe two similar and similarly placed polygons.

What point is the centre of similitude?

The ratio between what two lines represents the ratio between corresponding sides?

12. A circle being regarded as a polygon with an infinite number of sides, show that circles are similar figures.

The ratio between what lines represents the constant ratio between corresponding lines in two circles?

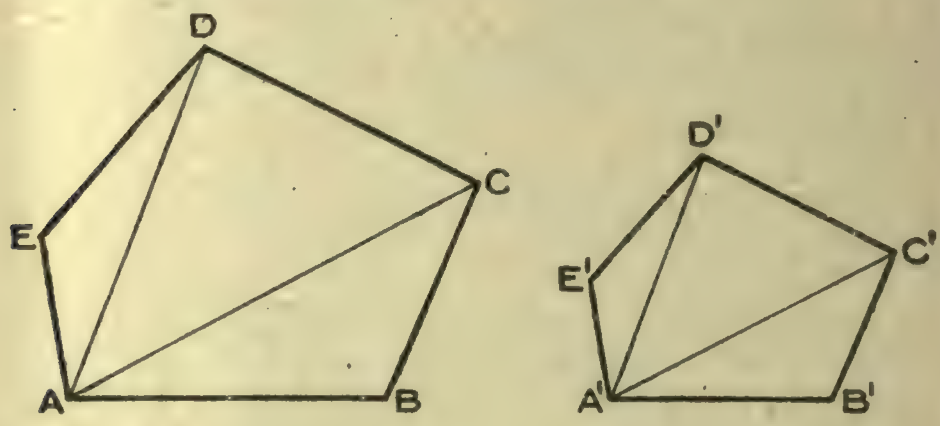
13. The ratio of the perimeters of two similar polygons is equal to the ratio of corresponding sides.

14. The ratio of the circumferences of two circles is equal to the ratio of their radii.

15. If AB, CD be two lines in the same straight line, and on AB a segment of a circle be constructed, use the principle of centre of similitude to construct for points on a segment on CD similar to that on AB.

PROPOSITION III. THEOREM.

ℳ **The areas of similar polygons are as the squares on corresponding sides.**



Let **ABCDE, A′B′C′D′E′** be two similar polygons.

Then their areas are as the squares on corresponding sides.

Join **AC, AD, A′C′, A′D′**; and let the ratio of corresponding sides $= m : n$.

Then since (V., 1, Cor. 1) the ratio of lines joining corresponding points $= m : n$, the $\triangle$s into which **ABCDE** has been divided are similar to the corresponding triangles into which **A′B′C′D′E′** has been divided. (III., 6.)

Hence $\triangle ABC : \triangle A'B'C' = AB^2 : A'B'^2 = m^2 : n^2$; (III., 11.)

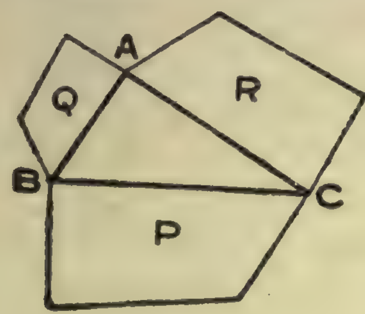
$$\triangle ACD : \triangle A'C'D' = CD^2 : C'D'^2 = m^2 : n^2;$$

$$\triangle ADE : \triangle A'D'E' = DE^2 : D'E'^2 = m^2 : n^2;$$

and $\therefore$ $ABDCE : A'B'C'D'E' = m^2 : n^2,$

$$= AB^2 : A'B'^2.$$

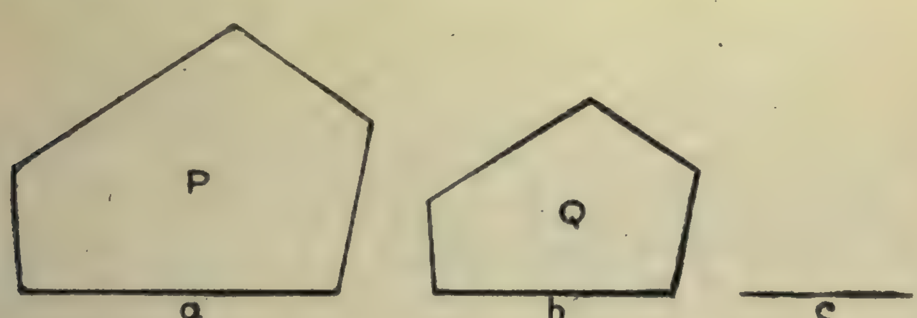
COR. 1. **If similar polygons be described on the sides of a right-angled triangle, the area of that on the hypotenuse is equal to the sum of the areas of those on the two other sides.**



$$\text{For } \frac{\text{Q}}{\text{P}} = \frac{\text{AB}^2}{\text{BC}^2}, \quad \frac{\text{R}}{\text{P}} = \frac{\text{AC}^2}{\text{BC}^2};$$

$$\therefore \ \frac{\text{Q}+\text{R}}{\text{P}} = \frac{\text{AB}^2 + \text{AC}^2}{\text{BC}^2} = 1.$$

COR. 2. If three straight lines, a, b, c, be proportionals, and similar polygons, **P**, **Q**, be described on the first and second, then



$$\frac{\text{P}}{\text{Q}} = \frac{a^2}{b^2} = \frac{a}{b} \cdot \frac{a}{b} = \frac{a}{b} \cdot \frac{b}{c} = \frac{a}{c}.$$

That is,—**If three straight lines be proportionals, as the first is to the third, so is any rectilineal figure described on the first to a similar and similarly described rectilineal figure on the second.**

Exercises.

1. Two similar polygons being given, construct a similar polygon equal to their sum.

Also one equal to their difference.

2. **The areas of circles are to one another as the squares of their radii.**

3. **The areas of segments of two different circles containing equal angles, are as the squares of the radii of the circles.**

4. **The areas of two sectors of circles having equal central angles, are as the squares of the radii.**

5. If on the perimeter of any polygon three points be taken, and on the perimeter of a similar polygon the three corresponding points be taken, and a triangle be formed in each polygon by joining these points, the areas of the triangles are as the areas of the polygons.

6. A trapezium whose parallel sides are a and c, is divided into two similar trapeziums by a line b parallel to a and c. Show that a, b, c are in continued proportion. Hence show that the two smaller trapeziums are equal in area respectively to the two triangles into which the whole trapezium is divided by either diagonal.

7. If four straight lines, a, b, c, d, be proportionals, the similar and similarly described rectilineal figures on a and b are to one another as the similar and similarly described figures on c and d.

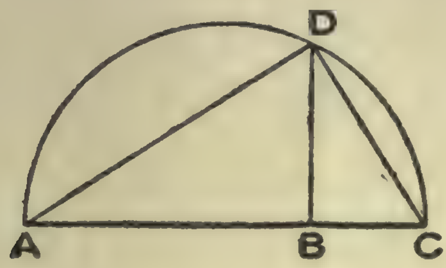
8. The triangle ABC has AB divided into four equal parts, and through the three points of division lines are drawn parallel to BC. Compare the areas contained between successive parallels.

9. Two similar polygons which are equal in area are equal in all respects.

PROPOSITION IV. PROBLEM.

HC

To find the mean proportional between two given straight lines.



Let **AB, BC** be the two given st. lines.

It is required to find the mean proportional between them.

Place **AB, BC** in a straight line, and on **AC** describe the semicircle **ADC**.

Through **B** draw **BD** at right angles to **AC**.

Then **BD** is the mean proportional between **AB** and **BC**.

Join **AD, DC**.

The ∠**ADC**, being in a semicircle, is a rt. ∠ (IV., 8), and **DB** is the ⊥r from the rt. ∠ to the base;

∴ **DB** is the mean proportional between **AB** and **BC**. (III., 12, Cor. 2.)

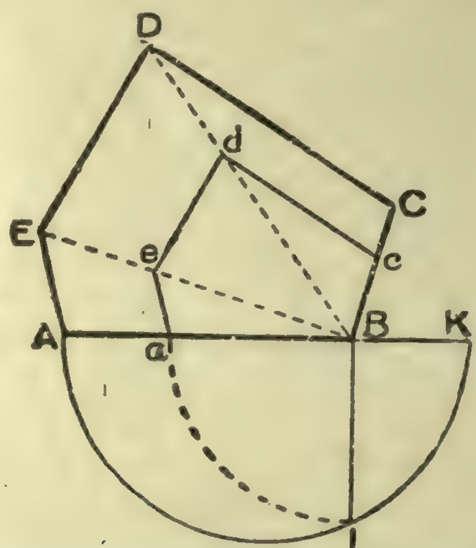
This problem is dealt with in Cor. 2, Prop. 12, Bk. III.; it is repeated here because it had not then been formally shown that the angle in a semicircle is a right angle.

As stated in Cor. 3, Prop. 12, Bk. III., the problem affords the means of describing a square equal to a given rectilineal figure. The general problem, however, of describing a rectilineal figure of *any* given shape and size will presently be dealt with.

In the Proposition, note that **AD** is the mean proportional between **AB** and **AC**; and that **CD** is the mean proportional between **CB** and **CA**.

PROPOSITION V. PROBLEM.

To construct a rectilineal figure similar to a given rectilineal figure, and such that the areas of the figures shall be in a given ratio.



Let **ABCDE** be the given rectilineal figure, and $m : n$ the given ratio.

It is required to construct a rectilineal figure similar to **ABCDE,** *and such that* **ABCDE** *shall be to it in the ratio* m : n.

In **AB** produced, take **BK,** such that **AB** : **BK** $= m : n$, as in Prop. 1, Bk. V.

On **AK** describe a semicircle **ALK,** and draw **BL** ⊥r to **AK**; so that **AB** : **BL** = **BL** : **BK.** (V., 4.)

Take **B**a = **BL,** and draw ae, ed, dc parallel to **AE, ED, DC** respectively.

Then a**B**cde is the rectilineal figure required.

For **ABCDE** and a**B**cde are similar by construction, **B** being the centre of similitude (V., 1), and

∴ **ABCDE** : a**B**cde = **AB**^{2} : a**B**^{2},

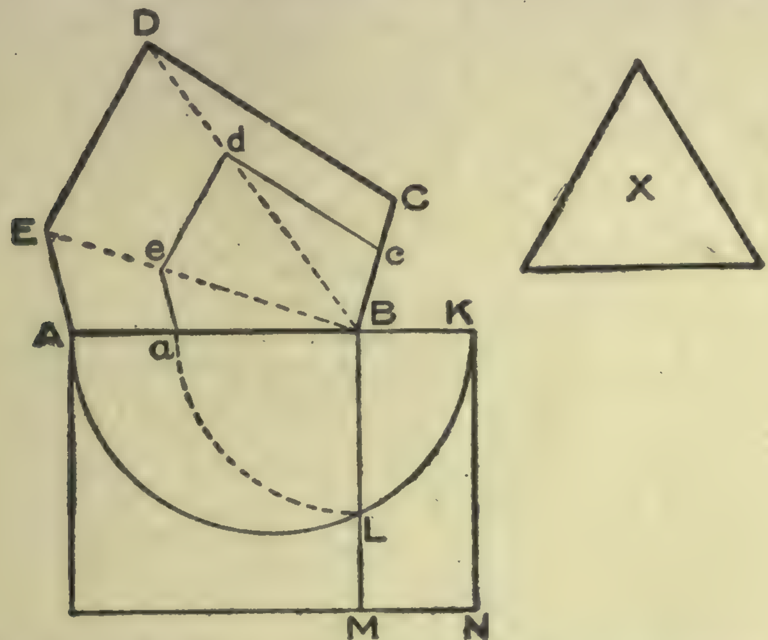
= **AB**^{2} : **BL**^{2},

= **AB** : **BK,** (shown in V., 3, Cor. 2.)

= $m : n$.

PROPOSITION VI. PROBLEM.

To describe a rectilineal figure which shall be similar to one and equal to another given rectilineal figure.



Let **ABCDE** and **X** be the two given rectilineal figures.

It is required to describe a figure similar to **ABCDE**, *and equal to* **X**.

On **AB** describe the rectangle **AM** = **ABCDE** (II., 12); and to **BM** apply the rectangle **BMNK** = **X** (II., 13).

On **AK** describe a semicircle **ALK**, cutting **BM** in **L**; so that **AB** : **BL** = **BL** : **BK**.

Take **B***a* = **BL**, and draw *ae*, *ed*, *dc* ∥ to **AE**, **ED**, **DC** respectively.

Then *a***B***cde* is similar to **ABCDE** and equal to **X**.

For since **ABCDE**, *a***B***cde* are similar by construction, **B** being the centre of similitude (V., 1.),

$$\therefore \mathbf{ABCDE} : a\mathbf{B}cde = \mathbf{AB}^2 : a\mathbf{B}^2,$$

$$= \mathbf{AB}^2 : \mathbf{BL}^2,$$

$$= \mathbf{AB} : \mathbf{BK} = \mathbf{AM} : \mathbf{BN} = \mathbf{AM} : \mathbf{X}.$$

But **ABCDE** is = **AM** by construction;

∴ *a***B***cde* is = **X**, and it is similar to **ABCDE**.

If instead of the area **X** being given in the form of a rectilineal figure, it be given as, say, 3 square inches, the rectangle **BMNK** is constructed of area 3 square inches, and the construction proceeds as before.

In familiar language the Proposition may be enunciated: **To describe a rectilineal figure of given size and shape.** It is evidently the general proposition of which "To describe a square equal to a given rectilineal figure" is a particular case.

Exercises.

1. Construct a square whose area shall be half that of a given square.

2. An angle of a parallelogram is 60°, and the sides containing it are 2 and 3 inches. Construct a similar parallelogram whose area shall be one-third that of the former.

3. Construct a pentagon and also a similar pentagon whose area shall be to that of the former as 2 : 3.

4. Construct a rectangle equal in area to a given square, and having its adjacent sides in the ratio 2 : 3.

5. Construct a triangle similar to a given triangle and equal in area to another triangle.

6. The triangle ABC has $A = 75°$, $AB = 2$, $AC = 3$ inches. Construct a triangle $A'B'C'$ equal in area to ABC and having $B' = 70°$, $C' = 55°$.

7. Construct a triangle ABC having $AB = 1$, $BC = 1\frac{1}{2}$, $CA = 2$ inches, and make an equilateral triangle equal to it in area.

8. Construct a triangle similar to ABC in Exercise 6, and equal to ABC in Exercise 7.

9. Construct a square equal to a given equilateral triangle.

10. Construct an equilateral triangle equal in area to a given square.

11. Construct an isosceles triangle equal in area to a given scalene triangle, and having a common vertical angle.

(Let ABC be scalene triangle. Construct isosceles triangle ABD with same vertical angle A. Let AE be mean proportional between AC, AD. Draw EF parallel to DB.)

12. Find two straight lines which have the same ratio to one another as two given rectilineal figures.

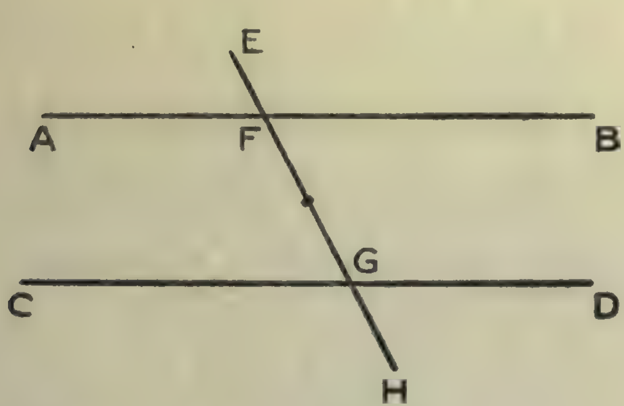
ADDITIONAL PROPOSITIONS.

Additional Propositions.

In the theory of parallels the three following propositions, with the accompanying definition and axiom, may be substituted for §§ 28–30 of the Introduction, Note 1, p. 80, being omitted.

DEF. Parallel straight lines are straight lines in the same plane, which do not meet however far they are produced in either direction.

PROP. 1. **If a straight line falling on two other straight lines, make the exterior angle equal to the interior and opposite angle on the same side of the line, the two straight lines are parallel to one another.**



Let the straight line **EH**, falling on the two straight lines **AB, CD**, make the exterior $\angle$**EFB** equal to the interior and opposite $\angle$**FGD.**

Then **AB** *is* $\parallel$ *to* **CD.**

For if $\angle$**EFB** = $\angle$**FGD**, then also (In., § 13) $\angle$**AFG** = $\angle$**FGD.**

Let now the figure rotate, through two right angles, about the middle point of **FG**.

Then **F** takes the place of **G**, and **G** of **F**.

Also because $\angle$**AFG** = $\angle$**FGD**, therefore **FA** coincides with the former position of **GD**, and **GD** with that of **FA**.

Hence the lines **AB**, **CD** coincide with the former positions of **DC**, **BA**.

Therefore if **AB**, **CD** meet, when produced, on one side of **EH**, they meet also, when produced, on the other side of **EH**.

But this is impossible. (In., § 8.)

Hence **AB**, **CD**, when produced, meet on neither side of **EH**; and they are therefore parallel.

NOTE. Evidently the first part of Prop. 3, Bk. II., has here been demonstrated, and therefore may subsequently be omitted.

PLAYFAIR'S AXIOM. Two straight lines which intersect cannot both be parallel to the same straight line.

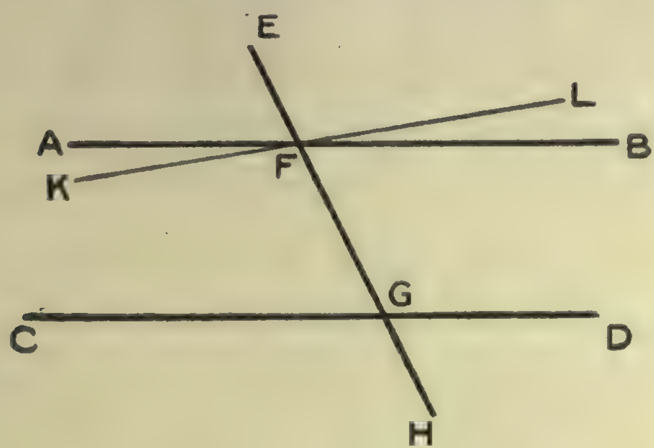
PROP. 2. **If a straight line fall on two parallel straight lines, it makes the exterior angle equal to the interior and opposite angle on the same side of the line.**

Let the straight line **EH** fall on the two parallel straight lines **AB**, **CD**.

Then the exterior $\angle$**EFB** *is equal to the interior and opposite* $\angle$**FGD**.

For if $\angle$**EFB** be not equal to $\angle$**FGD,** let $\angle$**EFL** be equal to $\angle$**FGD.**

Then (Prop. 1) **KFL** is parallel to **CD.**



That is, the straight lines **AB, KL** which intersect, are both parallel to the same straight line **CD**; which is impossible. (Playfair's Axiom.)

Hence $\angle$**EFB** = $\angle$**FGD.**

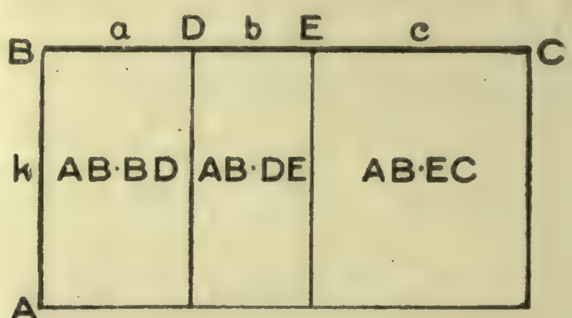
PROP. 3. **Straight lines which are parallel to the same straight line, are parallel to one another.**

For if they be not parallel they intersect; and therefore two intersecting straight lines are parallel to the same straight line, which is impossible. (Playfair's Axiom.)

Hence the straight lines are parallel.

The following five propositions afford geometrical illustrations of certain algebraic identities.

PROP. 4. **If there be two straight lines (AB, BC), one of which (BC) is divided into any number of parts (BD, DE, EC), the rectangle contained by the two straight lines is equal to the sum of the rectangles contained by the undivided line and the several parts of the divided line.**

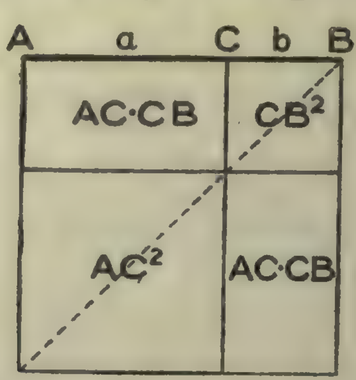


Evidently, $AB \cdot BC = AB \cdot BD + AB \cdot DE + AB \cdot EC$;

which is the algebraic identity,

$$k(a+b+c) = ka + kb + kc.$$

PROP. 5. **If a straight line (AB) be divided into any two parts (AC, CB), the square on the whole line is equal to the sum of the squares on the two parts, together with twice the rectangle contained by the parts.**

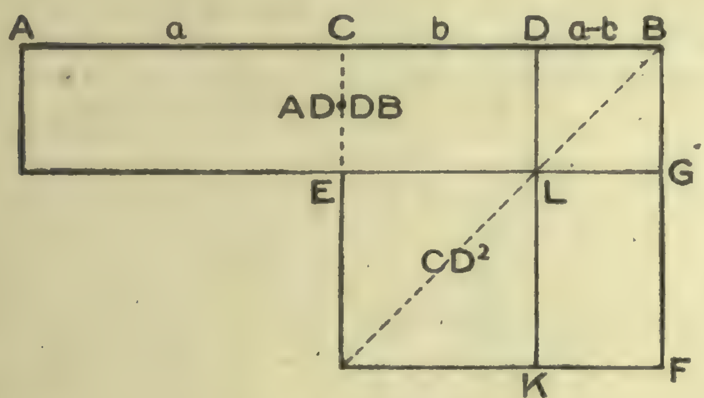


Evidently, $AB^2 = AC^2 + CB^2 + 2AC \cdot CB$;

which is the algebraic identity,

$$(a+b)^2 = a^2 + b^2 + 2ab.$$

PROP. 6. **If a straight line (AB) be divided into two equal parts (AC, CB), and also into two unequal parts (AD, DB), the rectangle contained by the unequal parts, together with the square on the line (CD) between the points of section, is equal to the square on half the line.**



The rectangle **AE** is equal to the rectangle **DF**, and ∴ the rectangle **AL** is equal to the figure **CGK**.

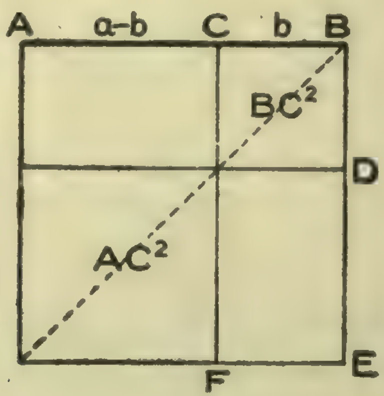
Hence $AD \cdot DB + CD^2 = CB^2$;

which is the algebraic identity,

$$(a+b)\ (a-b) + b^2 = a^2.$$

The same identity will also be found to be the equivalent of the proposition,—**If a straight line be bisected and produced to any point, the rectangle contained by the whole line thus produced, and the part of it produced, together with the square on half the line bisected, is equal to the square on the line made up of the half and the part produced.**

PROP. 7. **If a straight line (AB) be divided into any two parts (AC, CB), the sum of the squares on the whole line and on one of the parts is equal to twice the rectangle contained by the whole line and that part, together with the square on the other part.**



Evidently each of the rectangles **AD, CE** is the rectangle **AB . BC**; so that the figure **ADF** with **CD** is twice the rectangle **AB . BC.**

$$\text{Hence } AB^2 + BC^2 = 2AB \,.\, BC + AC^2;$$

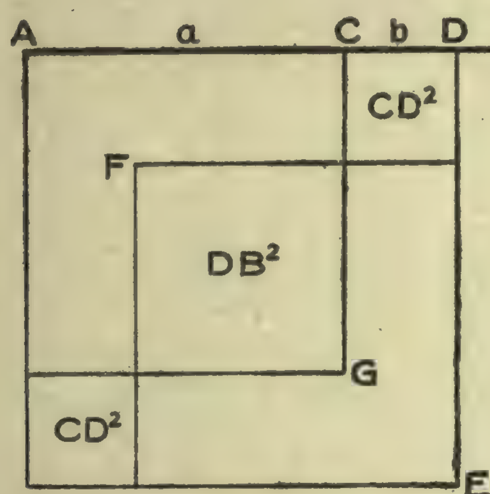
which is the algebraic identity,

$$a^2 + b^2 = 2ab + (a-b)^2,$$

$$\text{or } (a-b)^2 = a^2 - 2ab + b^2,$$

AB being a.

PROP. 8. **If a straight line (AB) be divided into two equal parts (AC, CB), and also into two unequal parts (AD, DB), the sum of the squares on the two unequal parts is double the sum of the squares on half the line and on the line (CD) between the points of section.**



$$EF = AG = AC^2.$$

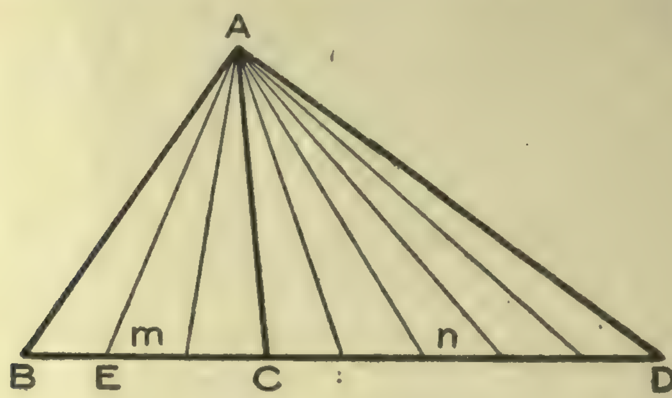
$$\text{Hence } 2AC^2 + 2CD^2 = AD^2 + DB^2;$$

which is the algebraic identity,

$$2a^2 + 2b^2 = (a+b)^2 + (a-b)^2.$$

The same identity will also be found to be the equivalent of the proposition,—**If a straight line be bisected and produced to any point, the square on the whole line thus produced, and the square on the part of it produced, are together double of the square on half the line bisected and of the square on the line made up of the half and the part produced.**

PROP. 9. Triangles of the same altitude are to one another as their bases.



Let the Δs **ABC, ACD** have the same altitude, namely the $\perp$r from **A** on **BD**.

Then the Δ**ABC** *is to the* Δ**ACD** *as* **BC** *to* **CD**.

Let **BE** be a common measure of **BC** and **CD**; and let it be contained m times in **BC**, and n times in **CD**.

Divide **BC** into m equal parts, each = **BE**, and **CD** into n equal parts, each = **BE**.

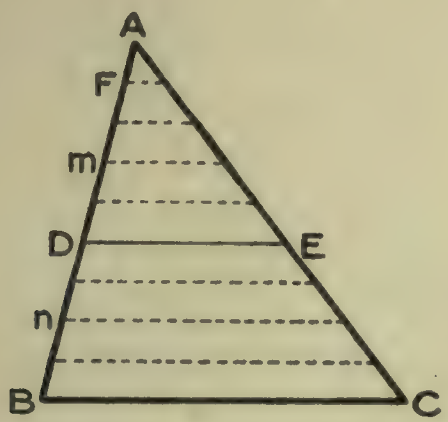
Form Δs by joining **A** to the points of division of **BD**. These Δs are all equal, since they are on equal bases and between the same $\|^s$; and there are m of these Δs in Δ**ABC**, and n in Δ**ACD**.

$$\therefore \Delta ABC : \Delta ACD = m : n = BC : CD.$$

Prop. 1, Bk. III., is also a demonstration of this theorem. The preceding proof may be substituted for that given on page 111.

PROP. 10. If a straight line be drawn parallel to one side of a triangle, it cuts the other sides proportionally.

And conversely, if a straight line cut two sides of a triangle proportionally, it is parallel to the third side.



Let **DE** be ∥ to **BC**, a side of the △**ABC**.

Then **AD** *is to* **DB** *as* **AE** *is to* **EC**.

Let **AF** be a common measure of **AD** and **DB**; and let it be contained m times in **AD**, and n times in **DB**.

Divide **AD** into m equal parts, each = **AF**, and **DB** into n equal parts, each = **AF**.

Through the points of division of **AD** and **DB** draw lines ∥ to **BC**. These lines intersect **AC** in points which divide it into equal parts, **AE** containing m of such, and **EC** containing n; (II., 5, Cor. 1.)

$$\therefore \mathbf{AD} : \mathbf{DB} = m : n = \mathbf{AE} : \mathbf{EC}.$$

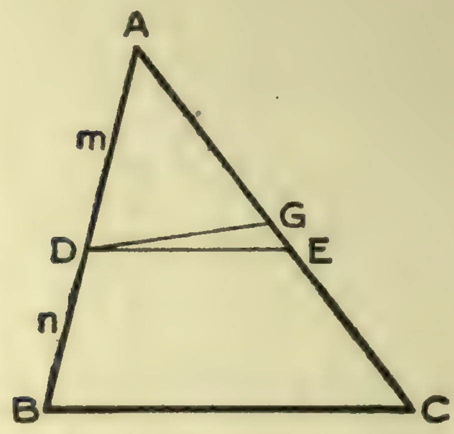
Evidently also,

$$\mathbf{AD} : \mathbf{AB} = m : m + n = \mathbf{AE} : \mathbf{AC}.$$

Conversely,—

Let **AD** be to **DB** as **AE** to **EC**.

Then **DE** *is* ‖ *to* **BC**.



For let **DG** be ‖ to **BC**.

Also let $AD : DB = m : n = AE : EC$;

then $AD : AB = m : m + n = AE : AC$.

Also since **DG** is ‖ to **BC**,

$AD : AB = AG : AC$;

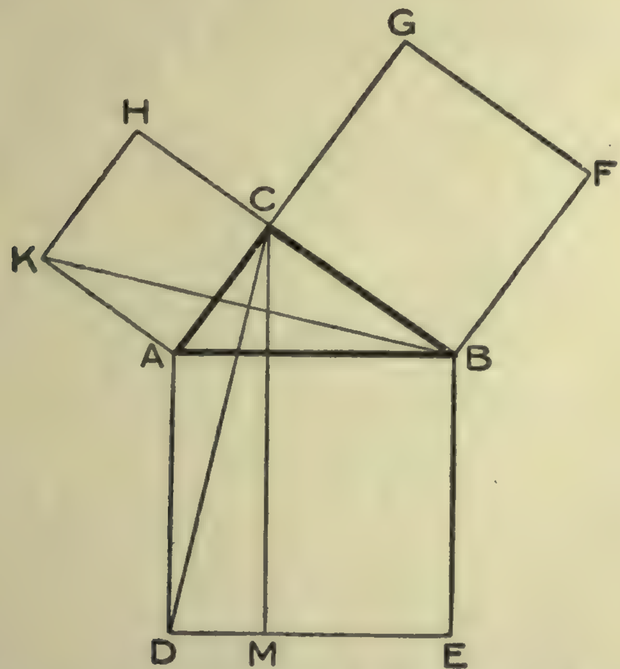
$\therefore AE = AG$.

Hence **E** and **G** coincide, and **DE** is ‖ to **BC**.

Prop. 2, Bk. III., is also a demonstration of this theorem. The preceding proof may be substituted for that given on pages 112–3.

The following is the usual figure and demonstration of Prop. 13, Bk. III. The proof here given may, of course, be substituted for that on page 140.

PROP. 11. **In a right-angled triangle, the square on the hypotenuse is equal to the sum of the squares on the sides containing the right angle.**



Let ABC be a right-angled △, having the rt. ∠ACB.

Then the square described on AB *is equal to the sum of the squares described on* AC, CB.

On AB, BC, CA describe the squares AE, BG, CK.

Through C draw CM ∥ to AD or BE.

Join CD, BK.

Because the ∠s ACH, ACB are right ∠s,

∴ HC, CB are in the same st. line.

Similarly AC, CG are in the same st. line.

Now rt. ∠KAC = rt. ∠DAB;

to each add ∠CAB;

∴ ∠KAB = ∠CAD.

Then in Δs **KAB, CAD,**

KA = **CA,**

AB = **AD,**

$\angle$ **KAB** = $\angle$ **CAD** ;

$\therefore$ Δ**KAB** = Δ**CAD.**

But sq. **CK** is double Δ**KAB,** since they are on same base and between same parallels.

For a similar reason $\|^m$ **AM** is double Δ**CAD.**

Hence sq. **CK** = $\|^m$ **AM.**

In the same way by joining **CE, AF,** it may be proved that

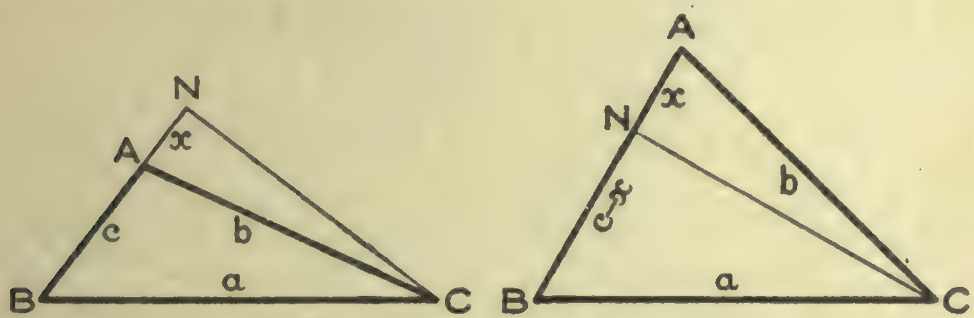
sq. **BG** = $\|^m$ **BM.**

But $\|^{ms}$ **AM, BM** make up the whole sq. **AE.**

Hence sq. **CK** + sq. **BG** = sq. **AE** ;

i.e., the square on **AB** is equal to the sum of the squares on **AC, CB.**

PROP. 12. **In any triangle, the square on the side opposite any angle, according as the angle is obtuse or acute, is greater or less than the sum of the squares on the sides containing the angle, by twice the rectangle contained by either of these sides and the projection upon it of the other.**



Let **ABC** be the Δ; and from **C** let **CN** be drawn $\perp$r to **BA**, so that **AN** is the projection of **AC** on **AB**.

Represent **BC** by a, **CA** by b, **AB** by c, and **AN** by x.

Then $a^2 = (c \pm x)^2 + \mathbf{CN}^2$, (III., 13.)

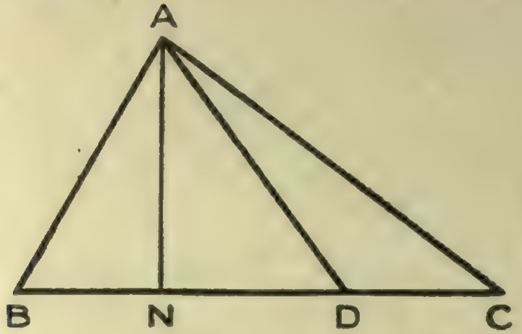
$$= (c \pm x)^2 + b^2 - x^2,$$

$$= c^2 \pm 2cx + x^2 + b^2 - x^2, \quad \text{(5 \& 7, Add. Props.)}$$

$$= b^2 + c^2 \pm 2cx.$$

Prop. 14, Bk. III., is also a demonstration of this theorem. The preceding proof may be substituted for that given on pages 144–5.

The following is an immediate deduction from the preceding proposition,—



ABC is any triangle having the segments of its base, BD : DC $= m : n$, or n BD $= m$ DC.

Then

$$AB^2 = AD^2 + BD^2 - 2\,BD\,.\,DN; \quad (1)$$
$$AC^2 = AD^2 + CD^2 + 2\,CD\,.\,DN. \quad (2)$$

Multiplying (1) by n and (2) by m, and adding, we have, since n BD $= m$ DC,

$$n\,AB^2 + m\,AC^2 = (m+n)\,AD^2 + n\,BD^2 + m\,CD^2.$$

If $m = n$, *i.e*, if BD = DC, then adding (1) and (2),

$$AB^2 + AC^2 = 2\,AD^2 + 2\,BD^2,$$

a result which occurs as an exercise on page 146.

Exercises.

1. ABCD is any rectangle, and E any point. Prove that $EA^2 + EC^2 = EB^2 + ED^2$.

2. If the squares on the sides of a quadrilateral are together equal to the sum of the squares on its diagonals, it must be a parallelogram. (It is easy to show that middle points of diagonals must coincide.)

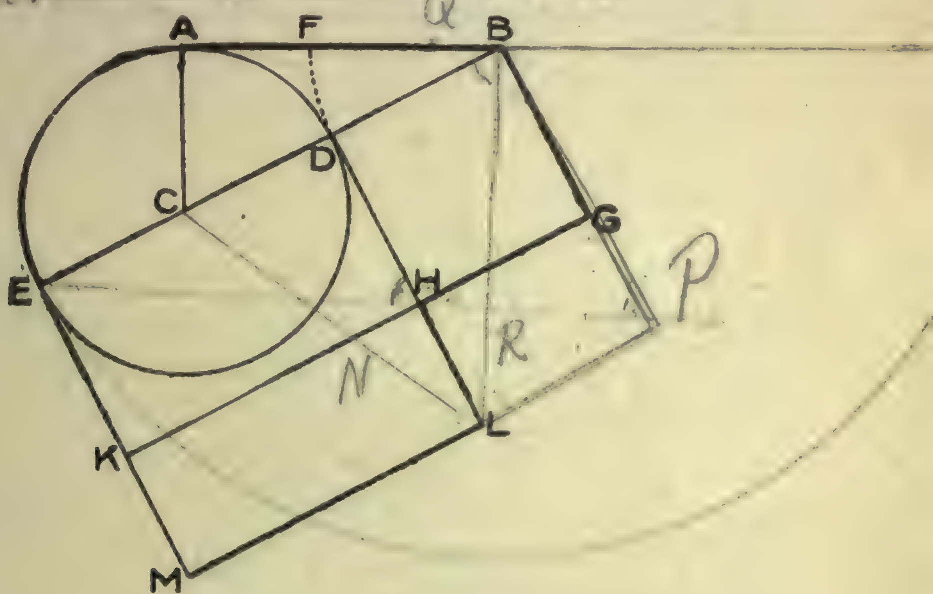
3. The squares on the equal sides of an isosceles triangle are together less than the squares on the two sides of any other triangle on the same base and having the same altitude.

4. On the side BC of any triangle ABC a square BDEC is described. Show that

$$DA^2 + AC^2 = EA^2 + AB^2.$$

For what quadrilateral BDEC, other than a square, will this equation hold?

PROP. 13. *To divide a straight line so that the rectangle contained by the whole line and one segment may be equal to the square on the other segment.* i.e. in medial section or in extreme and mean ratio



Let **AB** be the straight line to be divided.

Draw **AC** $\perp$r to **AB** and equal to one-half **AB**.

With centre **C** describe circle **EAD**. Join **BC**, meeting the circle in **D** and **E**; and from **BA** cut off **BF** = **BD**.

Then **BA . AF** = **BF**^{2}.

On **ED**, **DB** describe squares **EL**, **DG**; and produce **GH** to **K**.

Evidently **ED** = **AB**, and **HL** = **AF**.

Also, since **AB** is a tangent to the circle,

EB . BD = **AB**^{2};

$\therefore$ rect. **EG** = sq. **EL**;

$\therefore$ sq. **DG** = rect. **KL**;

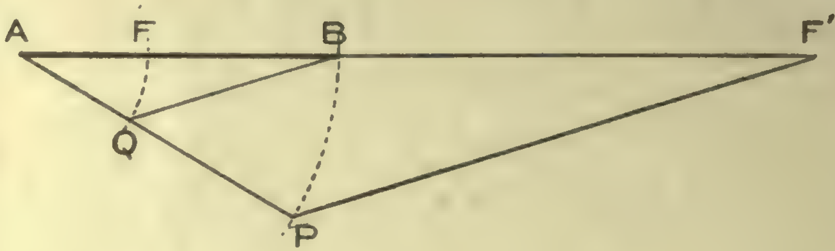
i.e., **BF**^{2} = **BA . AF**, and **AB** is divided at **F**, as required.

We have **BA** : **BF** = **BF** : **AF**, so that the whole line is to the greater segment as the greater segment to the less.

Such a line is said to be divided in **medial section.**

or in extreme and mean ratio

A straight line may also be divided *externally* in medial section.



For let **AP**(= **AB**) be divided in medial section in **Q**, so that $PQ^2 = PA \cdot AQ$. Join **BQ**, and let **PF′**, parallel to **QB**, meet **AB** produced in **F′**.

Then $$1 = \frac{PA \cdot AQ}{PQ^2} = \frac{PA}{PQ} \cdot \frac{AQ}{PQ},$$
$$= \frac{AF'}{BF'} \cdot \frac{BA}{BF'}; \quad \text{(III., 2.)}$$
or $BF'^2 = BA \cdot AF'$.

Thus **AB** is divided externally at **F′** into the segments **AF′** and **BF′**, so that the rectangle contained by the whole line **AB** and the segment **AF′** is equal to the square on the other segment **BF′**.

ALGEBRAIC EQUIVALENT :—Let $AB = a$, $BF = x$, and therefore $AF = a - x$. Then, since $BA \cdot AF = BF^2$, we have $a(a-x) = x^2$, or $x^2 + ax - a^2 = o$. The roots of this quadratic are $-\frac{a}{2} \pm \frac{a}{2}\sqrt{5}$, the positive root corresponding to **BF**, and the negative root to **BF′**.

Exercises.

1. If, in the figure of the Proposition, a circle be described with centre B, and radius BE, cutting AB produced in F′, then AB is divided externally in medial section at F′. [For BE . BD = BA²; ∴ BE (BE − BA) = BA²; ∴ BE² = BA² + BE . BA = BA (BA + BE); etc.]

In the figure of the Proposition prove the following:

2. BE is divided in medial section at D.

3. If CL be joined, cutting KG in N, then CL is divided in medial section in N.

4. If BG, ML produced meet in P, then E, H, P are in a straight line, and EP is divided in medial section in H.

5. If in FB, FQ be taken equal to FA, then FB is divided in medial section in Q. [BF² = BA . AF = (AF + FB) AF; ∴ BF (BF − AF) = AF²; etc.]

6. If EH produced meet BL at R, then ER is perpendicular to BL. [The Δs BDL, HDE are equal in all respects.]

7. Show that the lines DK and GL are both parallel to BM [D and K are corresponding points in the lines EB, EM. Similarly with L and G.]

8. Show that the rectangles BEKG, HKML are similar figures. [ED . DB = EH = EL − KL = ED² − DB² = (ED + DB) (ED − DB) = EB . KM; etc.]

9. Show that BK and HM are parallel.

10. Show that if CL, EH intersect in S, then S lies on the circumference of the circle. [∠CES = 90° − CBL (Ex. 6) = 90° − CLB (CL = CB) = LSH = CSE; etc.]

11. Show that DS (Ex. 10) is perpendicular to EH.

12. If BK cut DH in T, then HT = HL. [KT is parallel to HM.]

13. If BM cut HD, HK in U and V respectively, then BU = MV. [BM is parallel to GL.]

14. The rectangles BL, GM, DK are all equal.

15. If AB be produced to W, so that BW = BF, then AW . WB = AB².

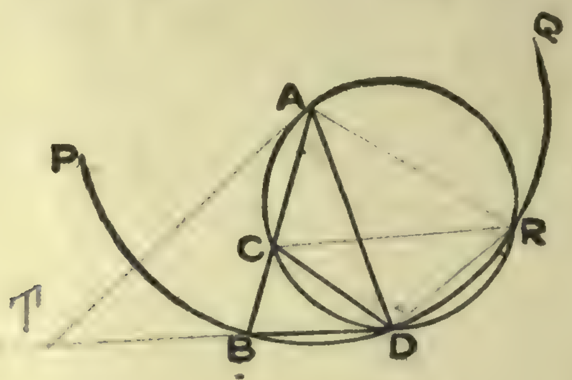
16. Show that BF² − AF² = AF . FB.

17. Show that AB² + AF² = 3BF².

18. Divide in medial section a line of length 60 millimetres, and verify the accuracy of your construction by calculation from the algebraic solution, and measurement.

19. Show that AB : BF = BF : AF

PROP. 14. *To describe an isosceles triangle having each of the angles at the base double of the third angle.*



Take any line AB, and divide it by C, so that $AB \cdot BC = AC^2$. (Add. Props., 13.)

With centre A, and radius AB, describe circle PBQ. In it place chord $BD = AC$. Join AD.

Then $\triangle$ ABD *has each of* $\angle$s ABD, ADB *double of* $\angle$BAD.

Join CD, and about $\triangle$ACD describe circle ACD.

Then $AB \cdot BC = AC^2 = BD^2$;

$\therefore$ BD touches circle ACD. (IV., 21.)

$\therefore \angle BDC = \angle CAD$, in alt. segment. IV, 13

To each add $\angle CDA$;

$\therefore \angle ADB = \angle CAD + \angle CDA$,

$= \angle BCD$;

and $\therefore$ also $\angle ABD = \angle BCD$;

$\therefore CD = BD = CA$;

$\therefore \angle CAD = \angle CDA$;

$\therefore \angle BCD$ is double of BAD;

and $\therefore$ each of $\angle$s ABD, ADB is double of $\angle$BAD.

Evidently $\angle$**BAD** $= \frac{1}{5}$ of $180° = 36°$; and $\angle$s **ABD, ADB** are each 72°.

We thus have the means of constructing, without the aid of a protractor, angles of 36°, 72°, 144°, 18°, 9°, etc.; or, as we may better express it, of taking $\frac{1}{5}, \frac{1}{10}, \frac{1}{20}, \ldots$ of a right angle.

We can also, by bisecting the angle of an equilateral triangle, take $\frac{1}{3}$, and therefore $\frac{1}{6}, \frac{1}{12}, \ldots$ of a right angle.

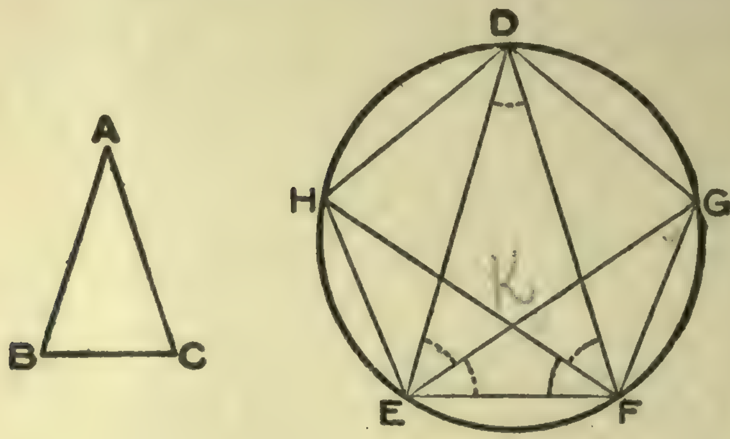
Since $\frac{1}{3} - \frac{1}{5} = \frac{2}{15}$, we can construct an angle which is $\frac{2}{15}$ of a right angle; and therefore angles which are $\frac{1}{15}, \frac{1}{30}, \frac{1}{60}, \ldots$ of a right angle.

We can also, without the use of a protractor, construct angles which are $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ of a right angle.

It has also been proved that a right angle can, by geometrical construction, be divided into any number of equal parts, provided such number is prime and of the form $2^n + 1$. This gives fractional parts,—$\frac{1}{3}, \frac{1}{5}, \frac{1}{17}, \frac{1}{257}, \ldots$

For the purposes of geometrical drawing, with its important bearing on the construction of machinery, the right angle can *approximately* be divided into any number of equal parts; and we must remember that even the straight line and circle can only approximately be constructed.

PROP. 15. *To construct a regular pentagon in a circle.*



Describe the △ABC, having each of the ∠s at the base double of the third angle.

In the given circle place the △DEF, equiangular to ABC. IV. 6

Bisect ∠s DEF, DFE by EG, FH, meeting the circle in G and H.

Join FG, GD, DH, HE.

Then DHEFG *is a regular pentagon.*

For the ∠s EDF, DEG, FEG, DFH, EFH, which are marked, are all equal to one another;

∴ the arcs and chords EF, DG, GF, DH, HE are equal to one another, and the pentagon is equilateral. IV. 9

Also arc HF = arc EG; and ∴ ∠HEF = ∠EFG. IV. 9

Similarly the other angles of the pentagon may be proved equal, and it is equiangular.

NOTE. A regular figure is one which has all its sides equal, and also all its angles equal.

Exercises.

1. What triangle in the figure of Prop. 14 has the vertical angle three times each angle at the base? ACD.

2. Divide a circle into two segments such that the angle in one is four times the angle in the other. What is the ratio of the lengths of the arcs of these segments? Line EF in fig. of Prop 15 – div

3. In the figure of Prop. 14, show that $\angle ADR = \angle ARD = 2\angle DAR$.

4. In the same, show that $DR = BD$.

5. In the same, show that the circle ACR is equal to the circle about the triangle ABD.

6. In the same, show that CR is parallel to BD, and bisects the angle ARD.

7. If the tangent to the small circle at A meet DB produced in T, then $BT = BA$.

8. In the figure of Prop. 14, show that $CR = AB$.

9. In the same, show that AC, CD, DR are the sides of a regular pentagon inscribed in the small circle.

10. Prove that the same three lines are equal to the sides of a regular decagon inscribed in the large circle.

11. In the figure of Prop. 14, prove that BR is the side of a regular pentagon inscribed in the large circle.

12. Construct an isosceles triangle, each of whose base angles is three-fourths of the vertical angle.

13. In the figure of Prop. 14, show that the centre of the arc CD is the centre of the circle about the triangle CBD.

14. How many degrees in the angle of a regular pentagon?

15. If the alternate angles of a regular pentagon be joined, show that they form another regular pentagon.

16. In the figure of Prop. 15, if EG, FH intersect in K, prove that GKF is an isosceles triangle with angles at base double of third.

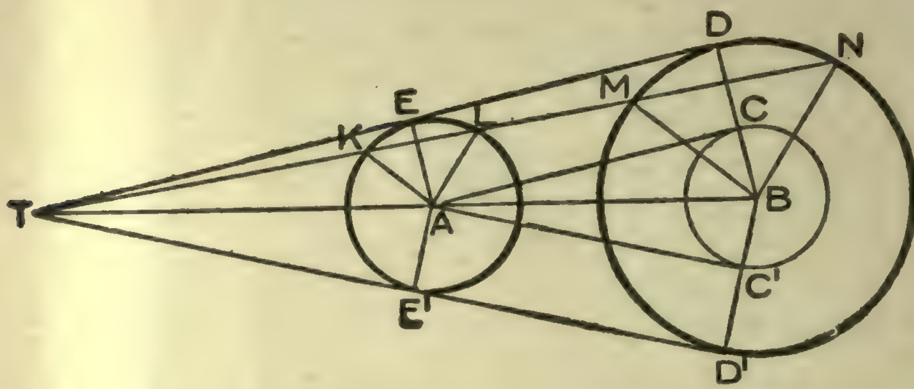
17. In the same, show that EG is divided in medial section at K.

18. In the same, show that HDGK is a rhombus.

19. In the same, show that EF is a tangent to the circle about the triangle GKF.

20. In the same, HE, GF are tangents to the circle about the triangle HKG.

PROP. 16. *To draw a common tangent to two given circles.*



Let A be the centre of the smaller circle, and B the centre of the larger.

With centre B, and radius equal to the difference of the radii, describe a circle CC′.

From A draw the tangent AC. Join BC and produce it to D. Draw AE ∥ to BD. Join ED.

Then ED *is a common tangent.*

For AE, CD are = and ∥;

∴ ACDE is a $\|^m$.

Also ∠C is a rt. ∠;

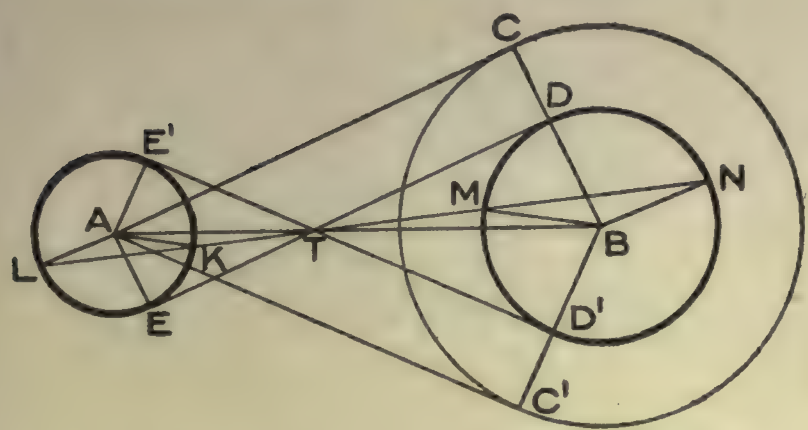
∴ other ∠s of the $\|^m$ are rt. ∠s;

∴ ED is tangent at both E and D.

Evidently another common tangent E′D′ exists on the other side of AB.

These tangents intersect BA in the same point T, for by similar △s TA : TB = AE : BD = AE′ : BD′, and T, E′, D′ are in the same st. line.

When the circles do not intersect, another pair of common tangents may be drawn, as shown in the following figure:



In this case, describe the circle **CC′** with centre **B** and radius equal to the sum of the radii of the given circles. The construction and proof proceed as before, except that **AE, BD** are on opposite sides of **AB**, and **T** falls between the circles.

In both cases **T** is a centre of similitude for the circles.

For draw the line **TKLMN** (**LKTMN** in second figure) cutting both circles. Then, as has already been pointed out, **TA** is to **TB** as the radii.

Hence in Δs **TAK, TBM**,

$$TA : TB = AK : BM,$$

and the $\angle$ **BTM** is common;

$\therefore$ the $\angle$s **TKA, TMB** are either equal or supplemental (III., 8). Evidently the angles **TKA, TMB** are equal; and the angles **TKA, TNB** are supplemental. For the angles **TKA, TMB** are both greater than right angles, since they fall within the triangles **TEA, TDB**, and therefore cannot be supplemental. Hence the angles **TLA, TNB**, being supplemental to the former, are less than right angles.

Hence the Δs **TAK, TBM** are similar; and also the Δs **TAL, TBN**;

$$\therefore \frac{TK}{TM} = \frac{AK}{BM},$$

a constant ratio, and **T** is a centre of similitude.

$$\text{Also } \mathbf{TE : TD = TL : TN};$$

$$\therefore \Delta\text{s } \mathbf{TEL, TDN} \text{ are similar.}$$

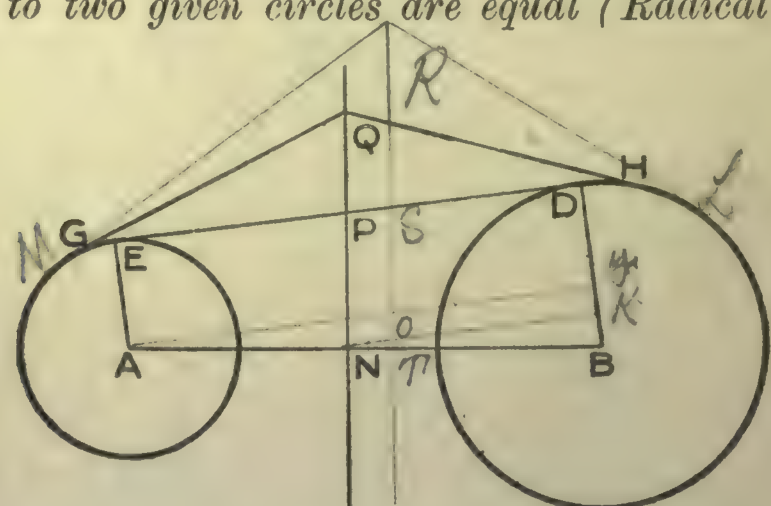
$$\text{And } \mathbf{TD^2 = TM . TN};$$

$$\therefore \frac{\mathbf{TM}}{\mathbf{TD}} = \frac{\mathbf{TD}}{\mathbf{TN}} = \frac{\mathbf{TE}}{\mathbf{TL}};$$

and **TM . TL = TD . TE,** and is constant.

In Exs. 6 and 7, page 205, attention was called to the fact that if two circles intersect, the tangents to the circles from any point on the chord are equal; also, conversely, that the locus of the point from which tangents are equal is the common chord. This locus was named the Radical Axis, and it was stated that when the circles did not intersect, the locus was still a straight line. We proceed to establish this:

PROP. 17. *To find the locus of the point from which tangents to two given circles are equal (Radical Axis).*



Draw **ED,** the common tangent to the two circles. 16
Bisect it in **P.** Then **P** is a point on the locus. Draw **PN** ⊥r to **AB,** and from any point **Q** in the fixed line **PN,** draw tangents **QG, QH** to the circles. These tangents are equal. For

Similarly $RM^2 = ES^2 + RT^2 - ST^2$
& $RL^2 = SD^2 + RT^2 - ST^2$

R being any ...
... GN
& RT ...

$$
\begin{aligned}
\mathbf{QG}^2 &= \mathbf{QA}^2 - \mathbf{AG}^2, \\
&= \mathbf{AN}^2 + \mathbf{QN}^2 - \mathbf{AE}^2, \\
&= \mathbf{AP}^2 - \mathbf{PN}^2 + \mathbf{QN}^2 - \mathbf{AE}^2, \\
&= \mathbf{AE}^2 + \mathbf{EP}^2 - \mathbf{PN}^2 + \mathbf{QN}^2 - \mathbf{AE}^2, \\
&= \mathbf{EP}^2 - \mathbf{PN}^2 + \mathbf{QN}^2.
\end{aligned}
$$

Similarly $\mathbf{QH}^2 = \mathbf{DP}^2 - \mathbf{PN}^2 + \mathbf{QN}^2$;

and $\mathbf{EP} = \mathbf{DP}$;

$\therefore \mathbf{QG} = \mathbf{QH}$.

It may readily be shown that tangents from any point not on **PN** are not equal. For the equality of **QG, QH** depended on the equality of **EP, DP,** and for a point not on **PN,** the perpendicular to **AB** would divide **ED** into unequal segments.

If the circles do not intersect, the easiest construction for the radical axis is to describe a third circle intersecting both those given. If **KL, MN** be the common chords, and **O** their intersection, **O** is a point on the radical axis, and the required line is the perpendicular from **O** on the line joining the centres of the given circles.

Exercises.

1. The radical axes of three circles, taken in pairs, meet in a point.

   This point is called the *radical centre* of the three circles.

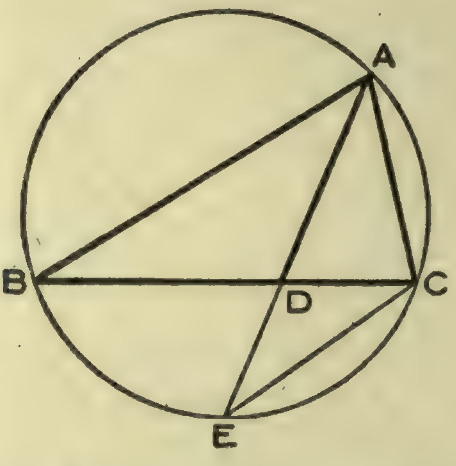
2. The difference between the squares on the tangents drawn from any given point to two given circles, is equal to twice the rectangle contained by the distance between the centres of the circles and the perpendicular from the given point on the radical axis.

3. The point of intersection of the perpendiculars from the angles of the triangle ABC on the opposite sides (called the orthocentre) is the radical centre of the circles described on the sides of ABC as diameters.

4. If, of three circles, each touches the other two, the common tangents at the points of contact are concurrent.

Ex 2, $EM^2 - PL^2 = ES^2 - SL^2 = (ES + SD)(ES - SL) = ED(EP + PS - EP$
$= 2\,EL \cdot PS$

PROP. 18. **If the vertical angle of a triangle be bisected by a straight line which also cuts the base, the rectangle contained by the sides of the triangle is equal to the rectangle contained by the segments of the base, together with the square on the straight line which bisects the angle.**



Let **ABC** be a $\triangle$, having $\angle$**BAC** bisected by **AD**.

Then $BA \cdot AC = BD \cdot DC + AD^2$.

Let a circle be described about $\triangle$**ABC**, and let **AD** produced meet the circumference in **E**.

Join **CE**.

Then in $\triangle$s **BAD**, **EAC**,

$$\angle BAD = \angle EAC;$$

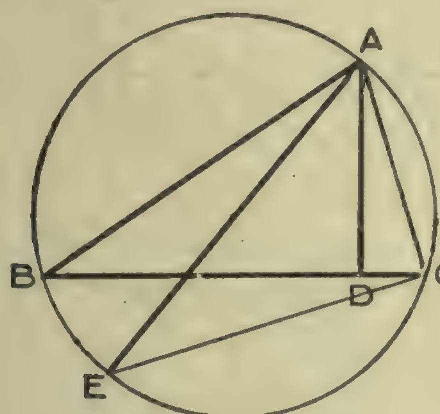
$$\angle ABD = \angle AEC; \quad \text{(IV., 6.)}$$

$\therefore$ $\triangle$s **BAD**, **EAC** are equiangular;

$$\text{and } \frac{BA}{AD} = \frac{EA}{AC};$$

$$\begin{aligned} \therefore BA \cdot AC &= EA \cdot AD, \\ &= (ED + DA)\, AD, \\ &= ED \cdot DA + AD^2, \\ &= BD \cdot DC + AD^2. \end{aligned}$$

PROP. 19. **If from the vertical angle of a triangle a straight line be drawn perpendicular to the base, the rectangle contained by the sides of the triangle is equal to the rectangle contained by the perpendicular and the diameter of the circum-circle.**



In the $\triangle$**ABC**, let **AD** be the perpendicular from **A** on **BC**, and let **AE** be the diameter of the circum-circle.

Then **BA . AC = EA . AD.**

Join **EC.**

Then in $\triangle$s **BAD, EAC,**

rt. $\angle$**ADB** = rt. $\angle$**ACE** in semicircle;

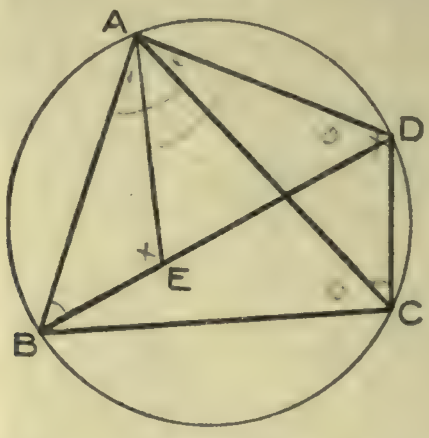
$\angle$**ABD** = $\angle$**AEC**; (IV., 6.)

$\therefore$ $\triangle$s **BAD, EAC** are equiangular;

and $\dfrac{\mathbf{BA}}{\mathbf{AD}} = \dfrac{\mathbf{EA}}{\mathbf{AC}}$;

$\therefore$ **BA . AC = EA . AD.**

PROP. 20. **The rectangle contained by the diagonals of a quadrilateral inscribed in a circle is equal to the sum of the two rectangles contained by its opposite sides.**



Let ABCD be a quadrilateral inscribed in a circle, and let AC, BD be its diagonals.

Then AC . BD = AB . CD + BC . AD.

Make $\angle$BAE = CAD.

To each of these add $\angle$EAC,

and $\angle$BAC = $\angle$EAD.

Then in $\triangle$s BAC, EAD,

$\angle$BAC = $\angle$EAD;

$\angle$ACB = $\angle$ADE; (IV., 6.)

$\therefore$ $\triangle$s BAC = EAD are equiangular;

and $\dfrac{AC}{BC} = \dfrac{AD}{DE}$;

$\therefore$ AC . DE = BC . AD.

Again in $\triangle$s BAE, CAD,

$\angle$BAE = $\angle$CAD;

$\angle$ABE = $\angle$ACD; (IV., 6.)

$\therefore$ $\triangle$s BAE, CAD are equiangular;

and $\dfrac{AC}{CD} = \dfrac{AB}{BE}$;

$$\therefore AC \cdot BE = AB \cdot CD.$$
$$\text{But } AC \cdot DE = BC \cdot AD;$$
$$\therefore AC(BE + DE) = AB \cdot CD + BC \cdot AD;$$
$$\text{or } AC \cdot BD = AB \cdot CD + BC \cdot AD.$$

Exercises.

√ 1. If AD bisect the angle BAC of a triangle ABC, and meet BC in D, then $(BA + BD)(CA - CD) = AD^2 = (BA - BD)(CA + CD)$.

2. Construct a triangle having given the base, the vertical angle and the rectangle contained by the sides.

√ 3. The exterior angle at A, of a triangle ABC, is bisected by AD′, meeting the base in D′. Show that $BA \cdot AC = BD' \cdot D'C - AD'^2$. [Let D′A meet the circum-circle in E. Then △s BAD′, EAC are similar.]

√ 4. If the diagonals of a quadrilateral inscribed in a circle be at right angles, the sum of the rectangles contained by opposite sides is equal to twice the area of the quadrilateral.

√ 5. A circle is described about an equilateral triangle, and from any point on the circumference straight lines are drawn to the angular points of the triangle; show that one of these lines is equal to the sum of the other two.

√ 6. If a quadrilateral be inscribed in a circle, and perpendiculars be dropped from any point on the circumference to opposite sides, and also to diagonals, the product of the perpendiculars to the opposite sides is equal to the product of the perpendiculars to the diagonals.

√ 7. ABC is an isosceles triangle, and on the base BC, or base produced, a point D is taken; show that the circles about the triangles ABD, ACD are equal.

√ If the triangle ABC be not isosceles, show that the diameters of the circles about the triangles ABD, ACD are as the sides AB, AC.

8. The rectangle contained by the diagonals of a quadrilateral is less than the sum of the rectangles contained by its opposite sides, unless a circle can be circumscribed about the quadrilateral. [ABCD the quadl. At A and B make ∠s BAE, ABE equal to ∠s CAD, ACD. Then △s ABE, ACD are equiangular; and also △s ABC, AED. Whence $AB \cdot CD + BC \cdot AD = AC\,(BE + ED)$. In a cyclic quadrilateral E will fall on BD, since ∠s ABD, ACD are equal.]

9. AD, DC are equal arcs of a circle, and B is a movable point on the circumference. Show that if B does not lie on the arc ADC, the sum of the lines BA, BC bears to BD a fixed ratio; and if B lies on the arc ADC, the difference of the lines BA, BC bears to BD a fixed ratio.

10. If a, b, c be the sides of the triangle ABC and R the radius of the circum-circle, show that

$$R = \frac{abc}{4 \text{ area of } \Delta}.$$

$$\left[R = \frac{bc}{2 \text{ perp. on BC}} = \frac{abc}{2ap} = \text{etc.}\right]$$

11. The quadrilateral ABCD is bisected by the diagonal BD, and AB . AD = CB . CD. Show that the angles at A and C are either equal or supplemental.

12. ABCD is a quadrilateral inscribed in a circle, and is bisected by BD; show that AD . AB = CB . CD.

Note on Inscribed and Escribed Circles—Metrical Relations.

Let ABC be a triangle, and D, E, F the points where the inscribed circle touches the sides BC, CA, AB. Let the circle escribed to the side BC touch these sides in G, H and K, respectively. Represent the sides by a, b, c. Then AE = AF, etc.; also CG = CH, etc.; and AE = AF = $\frac{1}{2}$(AE + AF) = $\frac{1}{2}$(a + b + c − 2BD − 2CD) = $\frac{1}{2}$(a + b + c − 2a) = $\frac{1}{2}$(b + c − a). Similarly, BD = BF = $\frac{1}{2}$(a − b + c); CD = CE = $\frac{1}{2}$(a + b − c). AE = AF = s−a; BD = BF = s−b; CD = CE = s−c

Again, AH = AK = $\frac{1}{2}$(AH + AK) = $\frac{1}{2}$(AC + CG + AB + BG) = $\frac{1}{2}$(a + b + c). The symmetry of this result shows that the distance from each angle of the triangle to the points of contact (on the sides containing that angle) of the circle escribed to the opposite side is the same.

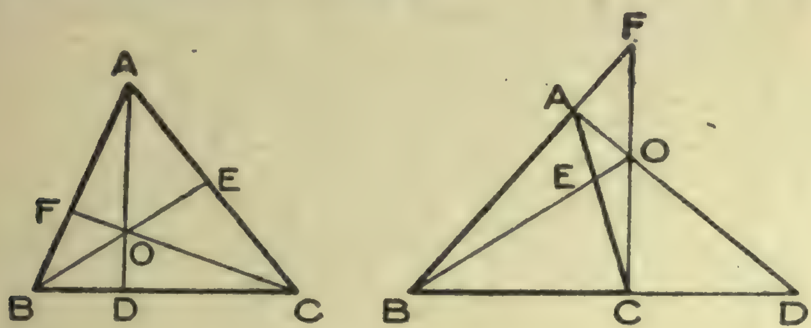
Also, CG = CH = AH − AC = $\frac{1}{2}$(a + b + c) − b = $\frac{1}{2}$(a − b + c). Similarly, BG = BK = $\frac{1}{2}$(a + b − c). CG = s−b; BG = s−c

Hence, CD = $\frac{1}{2}$(a + b − c) = BG; and CG = BD.

DG = CD − CG = $\frac{1}{2}$(a + b − c) − $\frac{1}{2}$(a − b + c) = b − c.

If L be the point where the circle escribed to AC touches BC, CL = $\frac{1}{2}$(a + b + c) − a = $\frac{1}{2}$(b + c − a). ∴ GL = CG + CL = $\frac{1}{2}$(a − b + c) + $\frac{1}{2}$(b + c − a) = c.

PROP. 21. **If three concurrent straight lines be drawn from the angular points of a triangle to meet the opposite sides, then the product of three alternate segments, taken in order, is equal to the product of the other three segments** (THEOREM OF CEVA).



Let **AD**, **BE**, **CF** be drawn from the angular points of a Δ through the point **O**, and cut the opposite sides in **D**, **E**, **F**.

$$\textit{Then}\ BD\,.\,CE\,.\,AF = DC\,.\,EA\,.\,FB.$$

$$\text{For } \frac{BD}{DC} = \frac{\triangle ABD}{\triangle ACD};\ \text{also } \frac{BD}{DC} = \frac{\triangle OBD}{\triangle OCD};$$

$$\therefore \frac{BD}{DC} = \frac{\Delta AOB}{\Delta COA}.$$

$$\text{Similarly } \frac{CE}{EA} = \frac{\Delta BOC}{\Delta AOB};$$

$$\text{and } \frac{AF}{FB} = \frac{\Delta COA}{\Delta BOC}.$$

Multiplying these ratios.

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1;$$

$$\text{or } BD\,.\,CE\,.\,AF = DC\,.\,EA\,.\,FB.$$

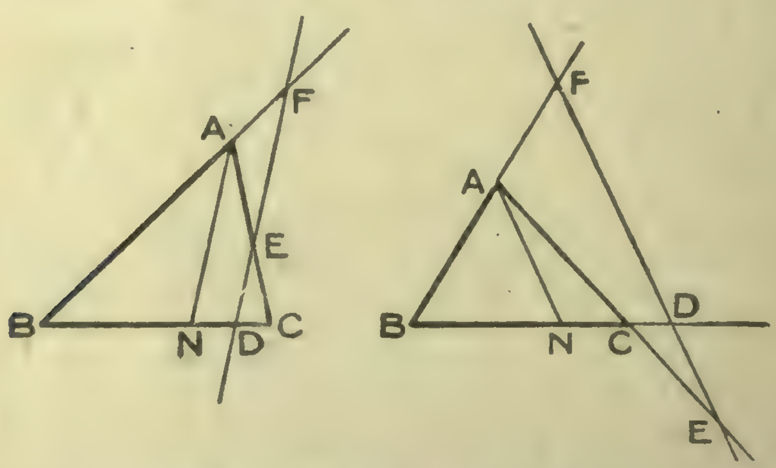
The converse of this theorem may be stated thus,— **If three straight lines, drawn from the angular points of a triangle, cut the opposite sides so that the product of three alternate segments,**

taken in order, is equal to the product of the other three, then the three straight lines pass through the same point.

It may readily be proved indirectly.

DEFINITION. A straight line which cuts a given system of lines is called a **transversal.**

PROP. 22. **If a transversal be drawn to cut the sides, or the sides produced, of a triangle, the product of three alternate segments, taken in order, is equal to the product of the other three segments.** (THEOREM OF MENELAUS.)



Let the straight line DEF cut the sides BC, CA, AB, of the $\triangle$ABC in D, E, F, respectively.

Then **BD . CE . AF = DC . EA . FB.**

Through A draw **AN** $\parallel$ to **DEF,** cutting **BC** in **N.**

Then by similar Δs,

$$\frac{BD}{BF}=\frac{ND}{AF};$$

$$\text{also } \frac{CE}{DC}=\frac{EA}{ND};$$

$$\therefore \text{ multiplying } \frac{BD}{BF}\cdot\frac{CE}{DC}=\frac{EA}{AF};$$

$$\text{or } BD\,.\,CE\,.\,AF=DC\,.\,EA\,.\,FB$$

The converse of this theorem may be stated thus,— **If three points be taken in two sides of a triangle and the third side produced, or in all three sides produced, and if the product of three alternate segments, taken in order, be equal to the product of the other three segments, the three points are in the same straight line.**

It may readily be proved indirectly.

Exercises.

1. If O be the orthocentre of the triangle ABC, it is also the radical centre of the circles described on OA, OB, OC as diameters.

2. The locus of a point which moves so that the difference of the squares on the tangents from it to two circles is constant, is a straight line parallel to the radical axis of the two circles.

3. If D, E, F be the middle points of the sides BC, CA, AB of the triangle ABC, show that AD, BE, CF are concurrent. (Theorem of Ceva.)

4. Show that the bisectors of the three angles of a triangle meet in a point.

5. Prove that the bisectors of two of the exterior angles of a triangle and of the remaining interior angle meet in a point.

6. The circle inscribed in the triangle ABC touches the sides BC, CA, AB in the points D, E and F, respectively. Show that AD, BE, CF are concurrent.

7. The three escribed circles of the triangle ABC touch the sides BC, CA, AB in the points D, E, F, respectively. Show that AD, BE, CF pass through a point.

8. In the triangle ABC, the lines AO, BO, CO meet the sides BC, CA, AB respectively in D, E, F; and the circle about DEF cuts the sides BC, CA, AB again in K, L, M, respectively. Show that AK, BL, CM are concurrent.

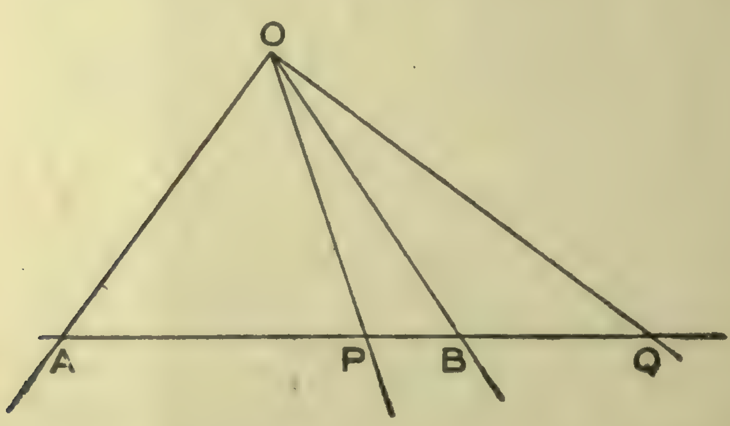
9. The circle inscribed in the triangle ABC touches the sides BC, CA, AB in D, E, F, respectively; and EF, FD, DE cut BC, CA, AB in K, L, M, respectively. Show that K, L, M are in the same straight line. (Theorem of Menelaus.) $\left(\dfrac{AF \cdot BK \cdot CE}{FB \cdot KC \cdot EA} = 1, \text{etc.}\right)$

Harmonic Ranges and Pencils.

When a number of points are *collinear*, *i.e.*, lie on the same straight line, they are said to form a **range.**

When a number of lines are *concurrent*, *i.e.*, pass through the same point, they are said to form a **pencil of rays,** or a **pencil.**

If the straight line AB be divided internally in P and externally in Q, in the same ratio, so that



$$\frac{AP}{PB} = \frac{AQ}{QB},$$

then A, P, B, Q form a **harmonic range.**

If O be an external point, the pencil OA, OP, OB, OQ, being concurrent lines through the points of a harmonic range, form a **harmonic pencil.**

Evidently,

$$AP \cdot BQ = AQ \cdot PB, \text{ and } \frac{AP \cdot BQ}{AQ \cdot PB} = 1,$$

forms in which the relation between the segments is often stated.

$$\text{Since } \frac{PB}{AP} = \frac{QB}{AQ};$$

$$\therefore \frac{AB - AP}{AP} = \frac{AQ - AB}{AQ},$$

$$\frac{AB}{AP} - 1 = 1 - \frac{AB}{AQ},$$

$$\frac{1}{AP} - \frac{1}{AB} = \frac{1}{AB} - \frac{1}{AQ}.$$

Hence $\frac{1}{AP}, \frac{1}{AB}, \frac{1}{AQ}$ are in Arithmetic Progression, and therefore **AP, AB, AQ** are in Harmonic Progression.

AB is said to be divided harmonically at **P** and **Q**; and **P, Q** are said to be the **harmonic conjugates** of **A, B.**

Since the relation $\frac{AP}{PB} = \frac{AQ}{QB}$ gives at once $\frac{PA}{AQ} = \frac{PB}{BQ}$, therefore **PQ** is divided harmonically at **A** and **B**; and **A, B** are the harmonic conjugates of **P, Q.**

PROP. 23. *To divide a given straight line internally and externally so that the segments may be in a given ratio, i.e., to divide a given straight line harmonically.*

Let **AB** be the given straight line, and $a : b$ the given ratio.

Through **A** and **B** draw two parallel lines **AX, BY** having the ratio $a : b$; and produce **YB** to **Y′** making **BY′ = BY.**

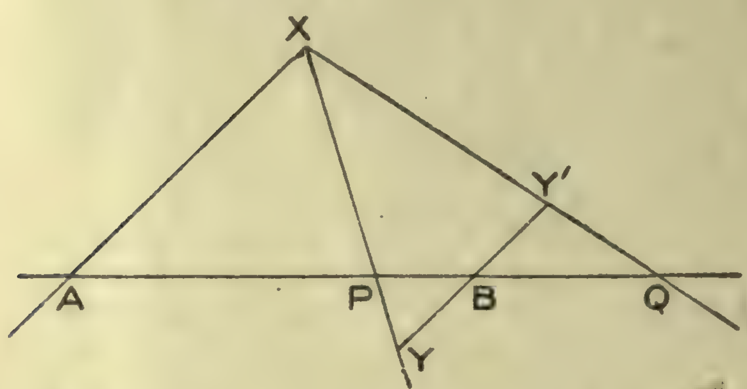
Join **XY, XY′**, cutting **AB** in **P** and **Q.**

$\frac{a}{b}$

Then $\frac{AP}{PB} = \frac{AX}{BY}$. (Sim. $\triangle$s **APX, BPY.**)

Also $\frac{AQ}{QB} = \frac{AX}{BY'}$, (Sim. $\triangle$s **XAQ, Y′BQ.**)

$= \frac{AX}{BY}$. $= \frac{a}{b}$



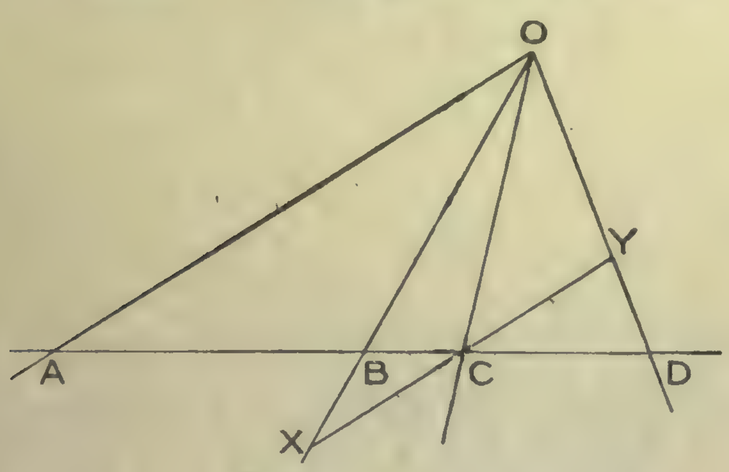
Hence $\frac{AP}{PB} = \frac{AQ}{QB}$, and **AB** is divided harmonically.

If three collinear points, **A, P, B,** be given, then taking **AX = AP** and **BY = BY′ = PB,** and constructing as in the proposition, a fourth point **Q** is found such that **A, P, B, Q** form a harmonic range.

It is important to observe that the solution is singular, *i.e.*, given three points **A, P, B,** in a straight line, only one point **Q** exists such that **A, P, B, Q** form a harmonic range.

Observe also that if $\frac{a}{b}$, the ratio of the segments, be infinite, **P** and **Q** coincide at **B.** As $\frac{a}{b}$ decreases, **P** and **Q** move in contrary directions from **B,** until, when $\frac{a}{b} = 1$, **P** becomes the middle point of **AB,** and **Q** is at infinity. When $\frac{a}{b}$ becomes less than 1, **Q** appears again on the left of A; and as $\frac{a}{b}$ decreases, P and Q move towards A, until when $\frac{a}{b} = 0$, they coincide with A. Thus as the ratio changes, P and Q always move in contrary directions; and as P moves from B to A, Q moves from B to the right to infinity, reappears from infinity at the left of A, and then moves up to A.

PROP. 24. **If A, B, C, D be a harmonic range, and therefore OA, OB, OC, OD a harmonic pencil; and if a line through C, parallel to OA, meet OB, OD in X and Y, respectively; then CX = CY.**

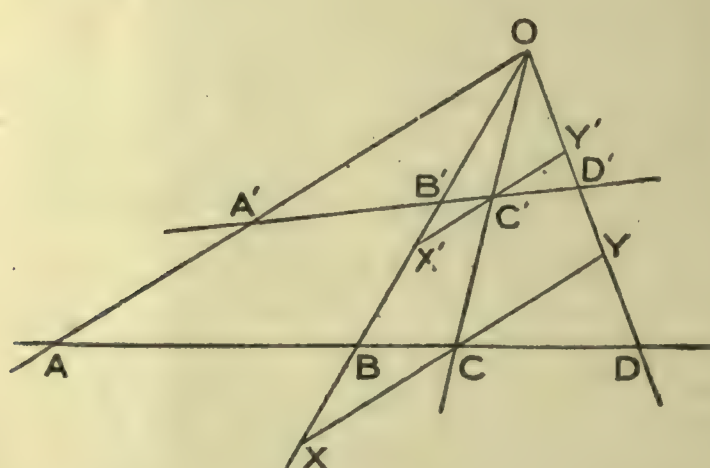


For $\frac{AB}{BC} = \frac{AD}{DC}$; but $\frac{AB}{BC} = \frac{AO}{CX}$, from similar $\triangle$s BAO, BCX; also $\frac{AD}{DC} = \frac{AO}{CY}$, from similar $\triangle$s DAO, DCY;

$\therefore$ CX = CY.

The construction for the harmonic division of a line (Prop. 23) would have suggested the truth of the preceding, since XA, XP, XB, XQ form a harmonic pencil.

Prop. 25. **If A, B, C, D be a harmonic range, and therefore OA, OB, OC, OD a harmonic pencil; and if another transversal cut the pencil in A′, B′, C′, D′, then A′, B′, C′, D′ form a harmonic range.**



Through **C** and **C′** draw **XCY**, **X′C′Y′**, parallel to **OA**. Then since **CX = CY** (Prop. 24), therefore **C′X′ = C′Y′**.

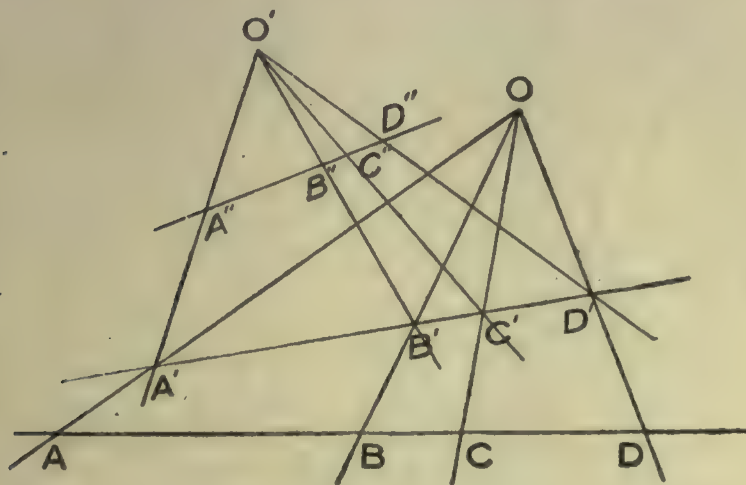
$$\text{Also } \frac{A'B'}{B'C'} = \frac{A'O}{C'X'}; \text{ (sim. } \triangle\text{s } B'A'O,\ B'C'X'.)$$

$$\text{and } \frac{A'D'}{D'C'} = \frac{A'O}{C'Y'}, \text{ (sim. } \triangle\text{s } OA'D',\ Y'C'D'.)$$

$$= \frac{A'O}{C'X'};$$

$\therefore \dfrac{A'B'}{B'C'} = \dfrac{A'D'}{D'C'}$, and **A′, B′, C′, D′** form a harmonic range.

Hence if we start with the harmonic range **A, B, C, D,** then **OA, OB, OC, OD** form a harmonic pencil; and therefore **A′, B′, C′, D′** form a harmonic range; and therefore



O′A′, O′B′, O′C′, O′D′ form a harmonic pencil; and therefore **A″, B″, C″, D″** form a harmonic range; and so on. Thus we may proceed from a harmonic range to a harmonic pencil, thence to a harmonic range, and so on indefinitely, by constructing pencils and drawing transversals, alternately.

The Complete Quadrilateral.

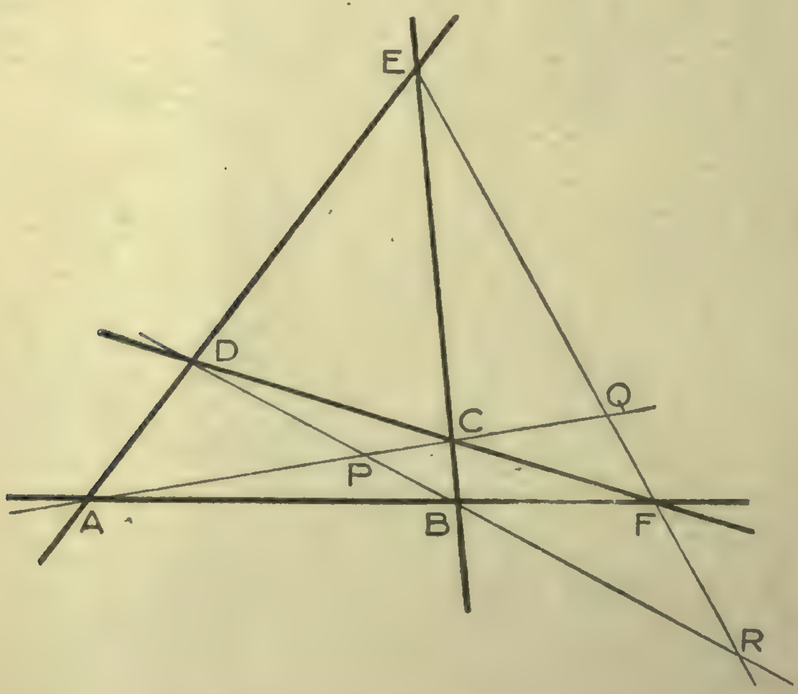
In Modern Geometry it is usual to think of the straight line as indefinitely extended both ways.

If we introduce this notion in connection with the triangle, it becomes a system of three infinite straight lines with their three points of intersection. Usually, in this generalized conception of the triangle, the lengths of the intercepted segments (the so-called sides of the triangle) and the area contained by them become of minor importance, or are not taken account of.

In like manner a quadrilateral becomes a system of four infinite straight lines, with their six points of intersection, since the four lines, taken two and two, meet in six points. Hence we have the following:

DEFINITION. **A system of four infinite straight lines, with their six points of intersection, is called a** COMPLETE QUADRILATERAL.

The four lines are called the **sides** of the quadrilateral. The six points of intersection are called the **vertices.**



Thus **AB, BC, CD, DA** are the sides; and **A, B, C, D, E, F** are the vertices.

Two vertices which do not lie on the same side are called **opposite vertices.**

Thus B and F lie on the same side with A; so also do D and E; C, however, does not; hence C is the vertex opposite to A. In like manner B and D are opposite vertices; so also are E and F.

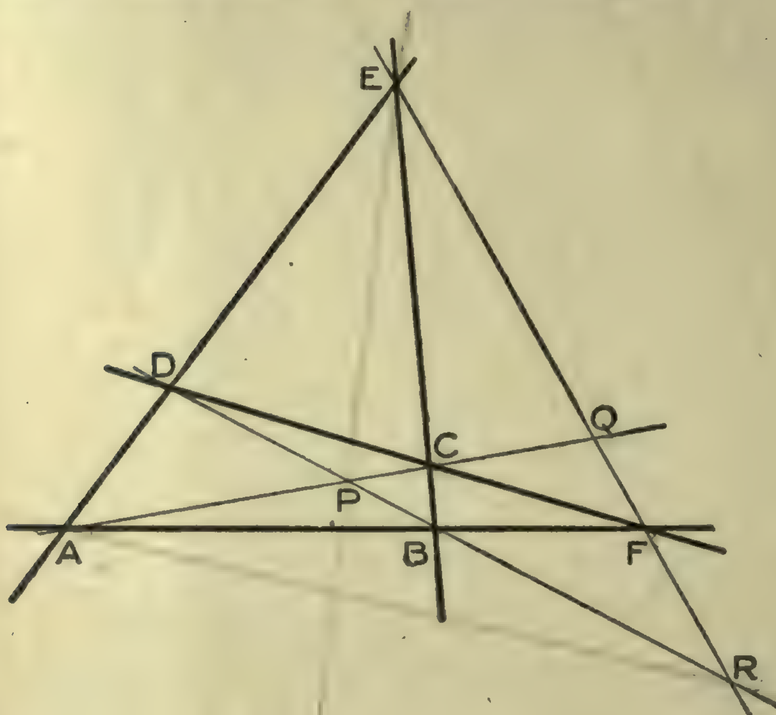
We may also consider the opposite vertices as the intersections of the pairs of sides in which, in three different ways, the sides may be arranged. Thus if **AB, BC** be a pair, **CD, DA** form the other pair, and **B, D** are opposite vertices. In like manner **AB, CD** and **BC, DA** give **F** and **E**; and **AB, DA** and **BC, CD** give **A** and **C**.

Again, if we take the sides in a particular order, in each case returning to the point from which we start, the opposite vertices are seen to be alternate points of intersection. Thus if the order be **AE, EC, CF, FA**, then **E** and **F** are alternate points of intersection, and also **C** and **A**; if the order be **DE, EB, BF, FD**, then **E** and **F** are alternate points of intersection, and also **B** and **D**; and if the order be **AD, DC, CB, BA**, then **D** and **B** are alternate points of intersection, and also **C** and **A**. The "opposite vertices" of the complete quadrilateral are thus seen to be a generalization of the "opposite angles" of the ordinary quadrilateral.

The three straight lines joining opposite vertices are called **diagonal lines,** or **diagonals.** The triangle formed by these diagonal lines is called the **diagonal triangle** of the complete quadrilateral.

Thus **AC, BD, EF** are the diagonals; and **PQR** is the diagonal triangle.

PROP. 26. **Opposite vertices of a complete quadrilateral, with the points in which the diagonal through them is cut by the two other diagonals, form a harmonic range.**



Consider the triangle **EAF**. Then since **EB, AQ, FD** pass through the point **C**, therefore

$$\frac{ED \cdot AB \cdot FQ}{DA \cdot BF \cdot QE}=1. \qquad \text{(Thm. of Ceva.)}$$

Also since **DBR** cuts the sides of the triangle **EAF**, therefore

$$\frac{ED \cdot AB \cdot FR}{DA \cdot BF \cdot RE}=1. \qquad \text{(Thm. of Menelaus.)}$$

Hence $\frac{FQ}{QE}=\frac{FR}{RE}$, *and* **R, F, Q, E** *form a harmonic range.*

Therefore **AR, AF, AQ, AE** form a harmonic pencil, *and* (Prop. 25) **R, B, P, D** *form a harmonic range.*

Therefore **ER, EB, EP, ED** form a harmonic pencil, *and* (Prop. 25) **Q, C, P, A** *form a harmonic range.*

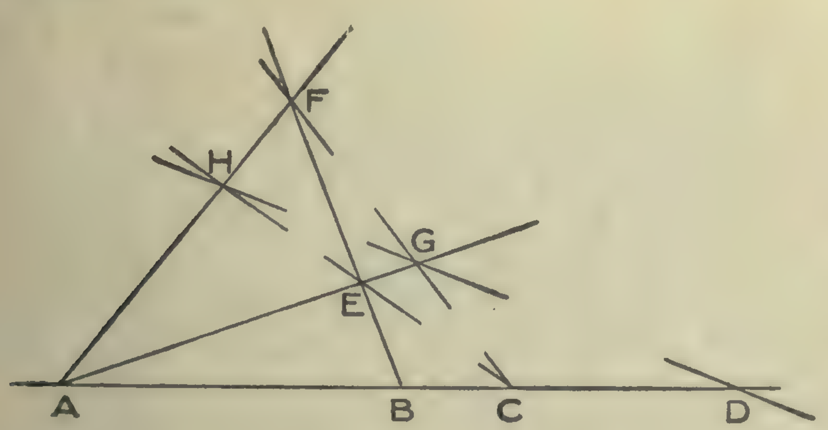
Δ EAC – point F – transversal DR
Δ ABC – „ E – „ DR

The Proposition is sometimes stated thus: **Each of the three diagonals of a complete quadrilateral is divided harmonically by the two other diagonals.**

The student may verify the following rule for the demonstration of the property in the case of a given diagonal:

Take the given diagonal and any two sides of the quadrilateral which do not intersect on this diagonal, to form the triangle required by the theorems of Ceva and Menelaus. Then the point in which the other two sides intersect is the point (Thm. of Ceva), and the other diagonal not through the intersection of the two sides first taken is the cutting line (Thm. of Menelaus).

The Proposition enables us, having given three points in a straight line, to find a fourth point, such that the four form a harmonic range, a straight ruler only being used in the construction, and no measurements being made.



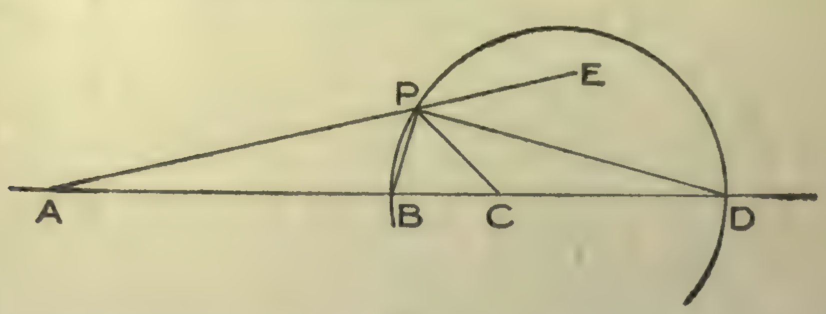
Thus let **A, B, C** be the three given points. Through **A** draw any two lines **AE, AF**; and through **B** draw any line cutting the former in **E** and **F**. Let **CE** cut **AF** in **H**, and **CF** cut **AE** in **G**. Then **HG** intersects **AC** in a point **D**, such that **A, B, C, D** form a harmonic range. For the quadrilateral **EGFH** has the diagonal **AC** intersected by the other diagonals in **B** and **D**.

With a view of familiarizing himself with this property, the student is advised to make several constructions for the point **D**, corresponding to three given points, **A, B,** and **C**, the lines **AE, AF, BEF** being taken in different positions. Great care will be found necessary in locating the points of intersection, **E, F, G, H**, that the same point **D** may always be reached.

Given also the points **C, D, A**, construct for **B**.

Also vary the position of **B** between **A** and **C**, and note the changes in the position of **D**.

PROP. 27. **If A, B, C, D be a harmonic range, and P be a point outside the line AD such that PA : PC = AB : BC = AD : DC, then the locus of P is a circle on BD as diameter.**



Since **PA : PC = AB : BC**, ∴ **PB** bisects ∠**APC**. Cnw, III, 3

Since **PA : PC = AD : DC**, ∴ **PD** bisects ∠**EPC**. [illegible]

Hence ∠**BPD** is a rt. ∠, and the locus of **P** is a circle on **BD** as diameter.

Exercises.

1. If A, P, B, Q be a harmonic range, and O be the middle point of AB, then $OA^2 = OP \cdot OQ$. [$AP \cdot BQ = AQ \cdot PB$; $\therefore (AO + OP)(OQ - OA) = (AO + OQ)(OA - OP)$; etc.]

2. Similarly, if O′ be the middle point of PQ, then $O'P^2 = O'A \cdot O'B$.

3. Conversely, if $OA^2 = OP \cdot OQ$, O not being between P and Q, then A, P, B, Q form a harmonic range. [$OA^2 = OP \cdot OQ$; $\therefore \frac{1}{4}(AP + PB)^2 = \frac{1}{2}(AP - PB) \cdot \frac{1}{2}(AQ + BQ)$; $\therefore (AP + PB)(AQ - BQ) = (AP - PB)(AQ + BQ)$; etc.]

4. In the figure of Prop. 27, show that the line joining the points of contact of tangents from A to the circle, passes through C.

5. Three points A, B, C being in the same straight line, find two points equidistant from C which divide the segment AB harmonically.

6. Find two points P and Q that are harmonic conjugates with respect to A and B, and also with respect to another pair of points, C and D. [Through A and B describe a circle, and describe another circle through C and D. If O be the point where their radical axis cuts AD, then $OA \cdot OB = OC \cdot OD$; etc. Note that C and D must not separate A and B; *i.e.*, the order of letters on the line must be A, B, C, D or A, C, D, B, not A, C, B, D. Note also that in the order A, C, D, B, the letters B and A are not separated, circular order being observed.]

7. If in a harmonic pencil the second ray bisect the angle between the first and third, then the fourth ray is at right angles to the second. Prove also the converse,—that if one pair of rays form a right angle, then they bisect internally and externally the angle between the others.

8. If straight lines be drawn through the intersection of the diagonals of a parallelogram parallel to the sides, they form with the diagonals a harmonic pencil. [Prop. 24.]

9. O, A, B, C and O, a, b, c **are harmonic ranges on intersecting lines with the point** O **common. Show that the lines** Aa, Bb, Cc, **joining corresponding points are concurrent.** Show also that Ca, Ac, Bb pass through a point. [Let Ca, Ac intersect in X; then XO, XA, XB, XC form a harmonic pencil; hence when BX is produced to cut Oc we get on Oc a harmonic range.]

10. **If two harmonic ranges are such that the straight lines joining three pairs of corresponding points pass through the same point, the line joining the fourth pair passes through the same point.**

11. **If two harmonic pencils are such that the intersections of three pairs of corresponding rays are on a straight line, the intersection of the fourth pair is on the same straight line.**

12. P, Q, R are collinear points on the sides BC, CA, AB of the triangle ABC, and P′, Q′, R′ are their harmonic conjugates with respect to these sides. Show that AP′, BQ′, CR′ are concurrent. [In quadrilateral QRBC diag. CR is divided harmonically by diagonals BQ, AP in, say, X and Y. Then AP′ passes through X, and BQ′ through Y. Also CB, CR, CA, CR′ form a harmonic pencil; and therefore P′, X, A, T form a harmonic range, if AP′, CR′ intersect in T. Again BP′, BX, BR, BY form a harmonic pencil; and therefore P′, X, A, T′ form a harmonic range if AP′, BQ′ intersect in T′; etc.]

13. ABC is a triangle, and the lines AX, BY, CZ, to the opposite sides, pass through a point. Show that if YZ meet BC in X′, then X′, B, X, C form a harmonic range. [Note diagonals of quadrilateral YZBC.]

14. If a circle touch the sides BC, CA, AB of a triangle in P, Q, R respectively, and QR meet BC produced in P′, then P, P′ are harmonic conjugates with respect to B, C.

15. **If** OA, OB, OC, OD **form a harmonic pencil, and** OA **be produced backwards to** A′, **show that** OB, OC, OD, OA′ **form a harmonic pencil. Hence if the lines** OB, OC, OD **be likewise produced backwards, any transversal in the plane is cut harmonically by these lines.**

16. When four points A, B, C, D, on a circle determine a harmonic pencil at any fifth point P on the circle, then if the point P move along the circumference, the lines PA, PB, PC, PD continue to form a harmonic pencil.

When P coincides with one of the four fixed points, what line forms the fourth ray of the pencil?

17. In the preceding exercise, if tangents be drawn at A, B, C, D, their intersections with the tangent at P form a harmonic range.

[Lines from centre to points of intersection make same angles with one another that PA, PB, PC, PD do.]

18. When four fixed tangents to a circle form by their intersections with a fifth a harmonic range, these four tangents determine a harmonic range on any tangent to the circle.

When the fifth tangent becomes one of the four, what four points form the range?

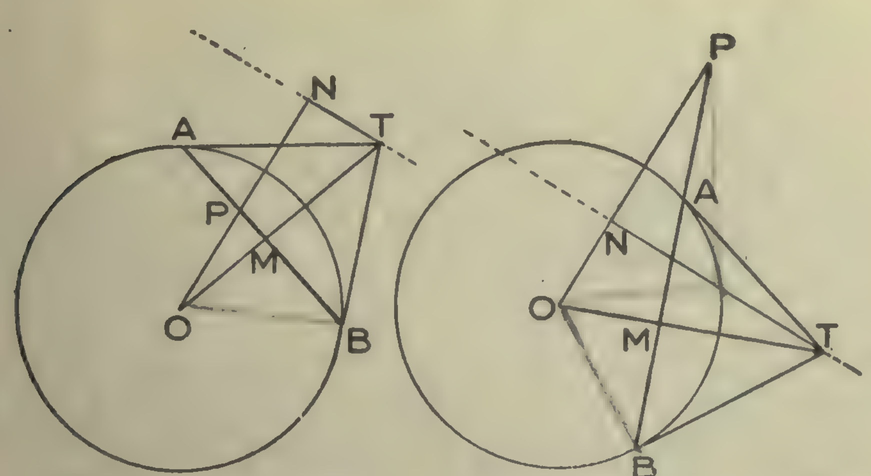
19. If two circles cut orthogonally, and AB, PQ be their diameters which are in the same straight line, then A, P, B, Q form a harmonic range. [Ex. 3.]

Poles and Polars.

The **Polar** of any point **P** with respect to a circle is the locus of the intersection of tangents drawn at the ends of any chord which passes through **P**.

The point **P** is called the **Pole** of the locus. It may be either within or without the circle; if it be on the circumference, the locus is evidently the tangent at **P**.

PROP. 28. **To find the polar of any given point P.**



Let **O** be the centre of the circle. Through **P** draw any straight line cutting the circle in **A** and **B**. Let the tangents at **A** and **B** intersect in **T**. From **T** draw a perpendicular **TN** on **OP**, or **OP** produced.

Join **OT** cutting **AB** in **M**. Then **BM** is perpendicular to **OT**. Hence, **OBT** being a right-angled triangle, and **BM** being the perpendicular on the hypotenuse, $OM \cdot OT = OB^2$. (Bk. III., 12, Cor. 1.)

Again, the angles at **M** and **N** being right angles, a circle may be described to pass through the points **P, M, T, N.** Therefore

$$OP \cdot ON = OM \cdot OT = OB^2 = (\text{radius})^2.$$

But the radius is constant, and **OP** is constant; therefore **ON** is constant.

Hence as **AB**, the chord through **P**, takes different positions, and in consequence the position of **T** varies, **N**, the foot of the perpendicular from **T** on **OP**, remains fixed. Therefore the locus of **T**, *i.e.*, the polar of **P**, is a straight line **TN** perpendicular to **OP** and passing through the fixed point **N**, which is determined by the relation $OP \cdot ON = (\text{radius})^2$.

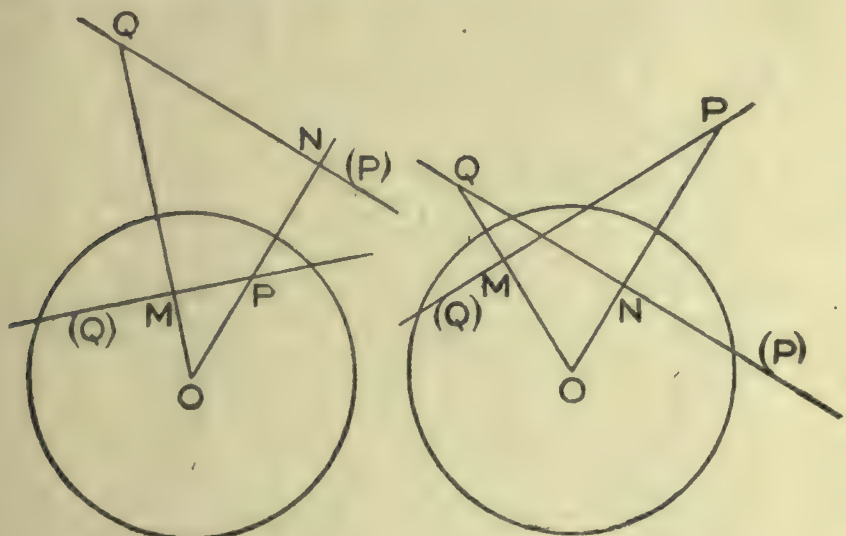
Hence having given the pole **P**, to construct the polar,—(1) if **P** be within the circle, draw a chord through **P**, perpendicular to **OP**, and at its end draw a tangent meeting **OP** in **N**: the line through **N**, perpendicular to **OP** is the polar; (2) if **P** be without the circle, draw a tangent from **P**: the line through the point of contact, perpendicular to **OP**, is the polar. These constructions suggest the method of constructing for the pole, when the polar is given.

It will be noted that when the pole is within the circle, the polar does not cut the circle; and when the pole is without the circle, the polar cuts the circle.

The polar of **P** may conveniently be denoted by enclosing **P** in brackets,—**(P)**.

Since the polar is at right angles to the line joining the centre to the pole, therefore the angle subtended at the centre by any two points P and Q is equal to one of the angles which the polars of P and Q make with one another.

u PROP. 29. **If Q lie on the polar of P, then P must lie on the polar of Q.**



Let (P) be polar of P, and let Q be any point on (P).

Then the polar of Q *passes through* P.

Join OQ, and draw PM perpendicular to OQ. Then the angles at M and N being right angles, a circle may be described to pass through the points P, M, Q, N.

Therefore $OQ \cdot OM = OP \cdot ON = (\text{radius})^2$.

Hence PM is the polar of Q, and P lies on the polar of Q.

u PROP. 30. **A chord of a circle is divided harmonically by any point on it and the polar of that point.**

Let AB be any chord of the circle through P; and let (P) be the polar of P, cutting AB in Q.

Then AB *is divided harmonically in* P *and* Q.

Let the tangents at **A** and **B** intersect on **(P)** at **T**; and let **N** be the intersection of **OP** with **(P)**.

Since the angles **OAT, OBT, ONT** are right angles, the circle described on **OT** as diameter passes through **A, B, O, T, N.**

In fig. 1 ∠ANO + ∠BNO stand on equal chords & ∴ on equal arcs of ⊙ AOBTN & they are at the O^ce ∴ ∠ANO = ∠BNO

Similarly in fig ∠ANT = ∠BN

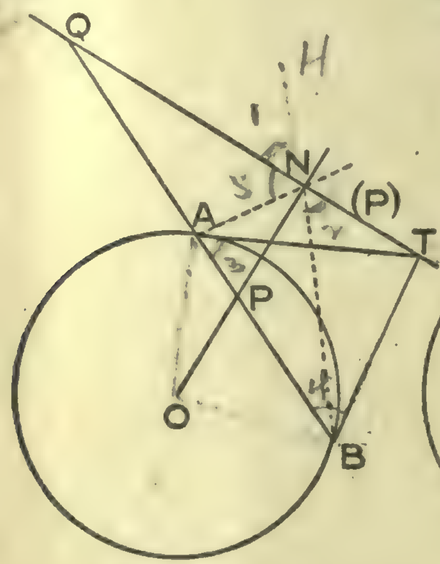


FIG. 1.

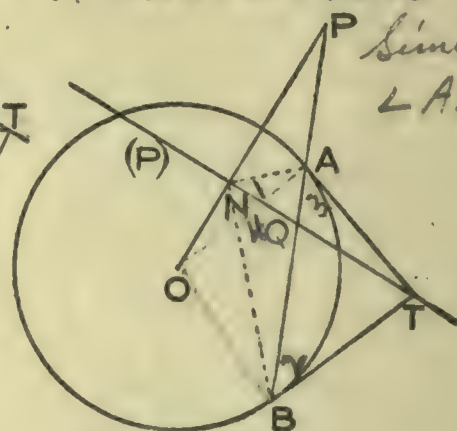


FIG. 2.

Also because **TA = TB**, therefore the angles **TNB, TNA** are supplemental in Fig. 1, and equal in Fig. 2.

Hence in Fig. 1, **NT** bisects the exterior angle of the triangle **ANB**, and the interior angle in Fig. 2; and therefore **NP**, which is perpendicular to **NT**, bisects the interior angle in the former case, and the exterior in the latter, and **AB** is divided harmonically in **P** and **Q**.

Exercises.

1. The straight line which joins two points P and Q is the polar of the intersection of the polars of P and Q. [For any line through P has its pole on (P), and any line through Q has its pole on (Q); therefore the pole of a line through both P and Q must be the intersection of (P) and (Q).]

2. The point of intersection of any two straight lines is the pole of the straight line joining their poles.

3. **What is the locus of the poles of all straight lines which pass through a given point?**

4. If four points form a harmonic range, their polars with respect to any circle form a harmonic pencil. [For the polars all pass through the pole of the line on which the range lies; and the straight lines joining the four points to the centre are inclined at the same angles as the polars of the points.] Prove converse.

5. If AB be any chord of a circle, and P, Q be harmonic conjugates with respect to A, B, then the polar of P passes through Q, and the polar of Q passes through P. [Follows at once from Prop. 30.]

6. A, B, C, D **are four points taken in order on the circumference of a circle.** AD, BC **intersect at** P; AC, BD **at** Q; **and** AB, CD **at** R. **Show that the triangle** PQR **is such that each vertex is the pole of the opposite side.** [Let BD meet PR in T. Then B, Q, D, T form a harmonic range, and RP, RC, RQ, RB a harmonic pencil. Hence if RQ cut AD in X and BC in Y, then P, D, X, A form a harmonic range, and also P, C, Y, B. But the polar of P cuts AD and BC harmonically. Therefore QR is the polar of P. Similarly QP is the polar of R; and therefore PR is the polar of Q.]

The triangle PQR, each of whose sides is the polar of the opposite vertex, is said to be **self-conjugate** with respect to the circle.

7. **Employ the preceding to draw tangents to a circle from a given point, using a ruler only, the centre of the circle not being known.**

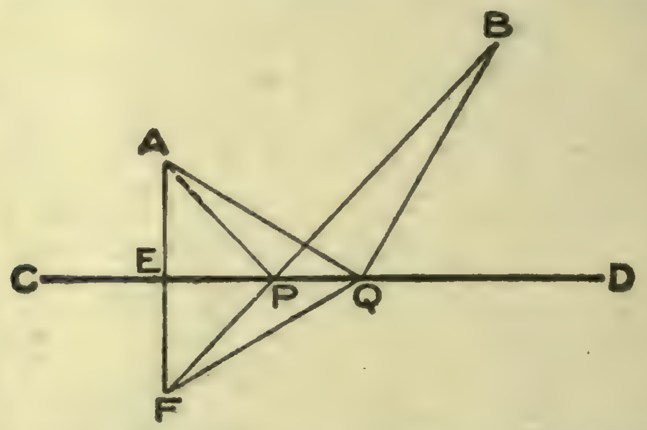
8. P, Q are any two points in the plane of a circle whose centre is C. PX is the perpendicular on the polar of Q, and QY the perpendicular on the polar of P. Show that PC . QY = QC . PX. [If PX, QY meet in R, then PCQR is a parallelogram. Draw perpendiculars CA, RB on PX, PC. Then PA . PR = PB . PC. Also CQ . CN = CP . CM, if CQ intersect (Q) in N, and CP intersect (P) in M; etc.]

9. If two circles cut orthogonally, and AB be any diameter of one of them, the polar of A with respect to the other circle passes through B.

PROP. 31. *To find a point on a given straight line, the sum of whose distances from two given points is the least possible.*

1. If the points be on opposite sides of the given line, the point required is evidently the intersection of the given line with the straight line joining the points.

2. Let the points be on the same side of the given line.



Let A, B be the two given points, and CD the given line.

From either of the points, say A, draw AE $\perp$r to CD. Produce AE to F, making EF = AE.

Join BF, cutting CD in P. Join AP.

Then AP + PB *is less than the sum of any other lines drawn from* A *and* B *to a point on* CD.

Let Q be <u>any other</u> point on CD.

Then points on CD are equidistant from A and F;

$$\therefore AQ + BQ = FQ + BQ,$$
$$> FB,$$
$$> FP + PB,$$
$$> AP + PB;$$

i.e., AP + PB is a minimum.

It will be noted that when AP + PB is a minimum, AP and BP make equal angles with CD.

Exercises.

1. Of all triangles on a given base and of given area, the isosceles triangle has the least perimeter.

2. Of all triangles on a given base and of given perimeter, the isosceles triangle has the greatest area.

3. If the perimeter of a triangle be given, the area is greatest when the triangle is equilateral. Use No 2

4. Two points A and B are on opposite sides of a line CD. Find a point P in CD, such that the difference between PA and PB may be a maximum. (CD bisects the angle APB.)

5. ABCD is an irregular quadrilateral. Show that, with the same perimeter and same diagonal BD, it is increased in area by converting it into a quadrilateral ~~A′B′C′D~~, with A′B = A′D and C′B = C′D.

6. Show that the quadrilateral of the preceding exercise, A′B C′D, with the same perimeter and the same diagonal A′C′, is increased in area by converting it into a rhombus A′B′C′D′.

7. Of all quadrilaterals with given perimeter, the square has the greatest area.

8. In exercise 4, if the points A and B are on the same side of CD, find P in CD, such that the difference between PA and PB may be a maximum.

9. Given the base and vertical angle of a triangle, show that its area is a maximum when the tangent to the circum-circle at the vertex is parallel to the base.

10. Of all triangles inscribed in the same circle, the equilateral has the greatest area.

11. From the angles A, B, C, of a triangle ABC, perpendiculars AD, BE, CF are drawn to the opposite sides; show that the triangle DEF has the least perimeter of all triangles with angular ~~points resting on~~ the sides of ABC. See Todhunter p.372 Deduction 360

12. Of all quadrilaterals inscribed in a given circle, the square has the greatest area. Use No 9

13. Of all quadrilaterals inscribed in a given circle, the square has the greatest perimeter. (Use Exercise 14, page 172.)

14. AB, AC are two intersecting lines, and D is a point between them. EDF is drawn terminated by AB and AC. Show that the area of the triangle AEF is a maximum when EF is bisected at D.